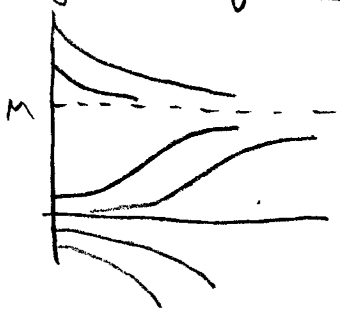
MTHSC 208, WEEK 5

Week 4 summary

- **Logistic equation**: $y' = ky\left(1 - \frac{y}{M}\right)$



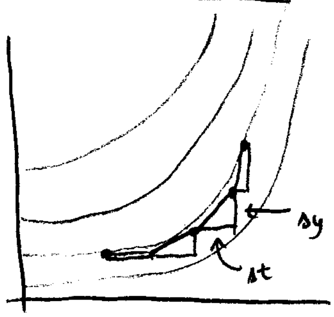
Carrying capacity M

Gen sol'n: $y(t) = \dfrac{M}{1 + Ae^{-kt}}$

initial: $y(0) = \dfrac{M}{1+A}$

limiting: $\lim_{t\to\infty} y(t) = M$.

- **Euler's method**: $y' = f(t, y)$



Given: (t_0, y_0) & step-size h. (i.e., $y(t_0) = y_0$)

Method: $(t_{k+1}, y_{k+1}) = (\underset{\Delta t}{t_k + h},\ y_k + \underbrace{h \cdot f(t_k, y_k)}_{\Delta y})$

- **2nd order ODEs**: $y'' = f(t, y, y')$

  ex: $y'' = k^2 y$ sol'n $y(t) = C_1 e^{kt} + C_2 e^{-kt}$

  $y'' = -k^2 y$ sol'n $y(t) = A\cos kt + B\sin kt$

- 2nd order linear ODEs have a two-parameter family of sol'ns:

  $y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$.

Question: Find the general sol'n to $y''-3y'-2y=0$

What might be a good guess?

Try $y(t)=e^{rt}$ where r is some const.

Solve for r.

$$y=e^{rt}$$
$$y'=re^{rt}$$
$$y''=r^2e^{rt}$$

Plug into $y''-3y'-2y=0$

$$r^2e^{rt}-3re^{rt}-2e^{rt}=0$$
$$e^{rt}(r^2-3r-2)=0$$
$$e^{rt}(r-1)(r-2)=0 \implies r=1,\ 2.$$

Thus, the sol'n is $\boxed{y(t)=C_1e^{t}+C_2e^{2t}}$

Question: What if we have a repeated root?

e.g., $y''-6y'+9y=0$

Again, guess
$$\left.\begin{aligned} y&=e^{rt}\\ y'&=re^{rt}\\ y''&=r^2e^{rt}\end{aligned}\right\}\quad \begin{aligned} &r^2e^{rt}-6re^{rt}+9e^{rt}=0\\ &e^{rt}(r^2-6r+9)=0\\ &e^{rt}(r-3)^2=0 \implies r=3\end{aligned}$$

Clearly $y(t)=C_1e^{3t}$ is a sol'n. But we need one more.

Try $y(t)=v(t)e^{3t}$, & solve for $v(t)$.

$y = ve^{3t}, \quad y' = 3ve^{3t} + v'e^{3t}$

$$y'' = 3(3ve^{3t} + v'e^{3t}) + 3v'e^{3t} + v''e^{3t}$$
$$= 9ve^{3t} + 6v'e^{3t} + v''e^{3t}$$

Plug back into ODE:

$$\underbrace{(\cancel{9ve^{3t}} + \cancel{6v'e^{3t}} + v''e^{3t})}_{y''} - 6\underbrace{(\cancel{3ve^{3t}} + \cancel{v'e^{3t}})}_{y'} + 9\underbrace{(\cancel{ve^{3t}})}_{y} = 0$$

$$v''e^{3t} = 0 \quad \Rightarrow \quad v'' = 0 \quad \Rightarrow \quad v'(t) = C$$
$$v(t) = Ct + D.$$

<u>Conclusion</u>: e^{3t} is a solution, <u>and</u> $(Ct+D)e^{3t}$ is a sol'n

We seek a fundamental pair (two truely distinct) of solutions.

Since any C & D will do, let's choose $C=1$, $D=0$.

$y_1(t) = C_1 e^{3t}$, $\quad y_2(t) = C_2 t e^{3t}$, so the general sol'n is

$$\boxed{y(t) = C_1 e^{3t} + C_2 t e^{3t}}$$

<u>Note</u>: Suppose we had chosen $C=5$, $D=10$. Then we would have $y_2(t) = C_2(5t+10)e^{3t}$, with general sol'n

$$y(t) = C_1 e^{3t} + C_2(5t+10)e^{3t}.$$

Convince yourself that is describes the same set of solutions.

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<u>Question</u>: What if the roots aren't real?

e.g. $y(t) = y'' + 2y' + 2y = 0$.

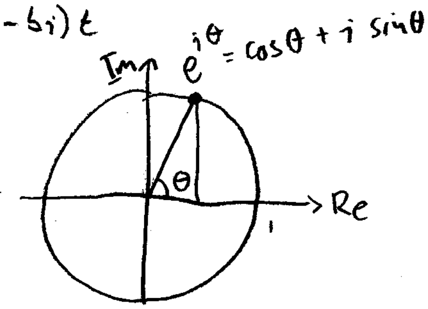
Guess $y(t) = e^{rt}$: $r^2 e^{rt} + 2re^{rt} + 2e^{rt} = 0$

$$e^{rt}(r^2 + 2r + 2) = 0$$

$$r = \frac{-4 \pm \sqrt{4-8}}{2} = -2 \pm i \quad \text{complex roots}$$

In general, suppose we get roots $\lambda = a + bi$
$\bar{\lambda} = a - bi$

Then we have 2 solutions: $e^{(a+bi)t}$, $e^{(a-bi)t}$



* Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Though the general solution is

$y(t) = C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t}$, there's a "better" way to write this.

$$e^{(a+bi)t} = e^{at}e^{bit} = e^{at}(\cos bt + i\sin bt)$$

$$e^{(a-bi)t} = e^{at}e^{-bit} = e^{at}(\cos(-bt) + i\sin(-bt))$$
$$= e^{at}(\cos bt - i\sin bt).$$

Now, we have sol'ns : $y_1(t) = e^{at}(\cos bt + i\sin bt)$
$y_2(t) = e^{at}(\cos bt - i\sin bt)$

<u>Problem</u>: Complex numbers are scary!

Recall: Since our ODE is linear & homogeneous, we can

- add two solutions
- multiply sol'ns by scalars

...and still have a sol'n.

Thus, $\frac{1}{2}\left(y_1(t)+y_2(t)\right) = e^{at}\cos bt$ is a sol'n

and $\frac{1}{2i}\left(y_1(t)-y_2(t)\right) = e^{at}\sin bt$ is a sol'n.

Conclusion: The general solution is

$$y(t) = Ae^{at}\cos bt + Be^{at}\sin bt$$

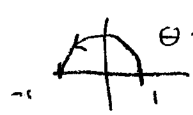
or $\boxed{y(t) = e^{at}\left(A\cos bt + B\sin bt\right)}$

Let's revisit **Euler's formula:** $\boxed{e^{it} = \cos t + i\sin t}$

Recall: $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$.

$$\left.\begin{aligned} e^{it} &= \cos t + i\sin t \\ e^{-it} &= \cos t - i\sin t \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \tfrac{1}{2}\left(e^{it}+e^{-it}\right) &= \cos t \\ \tfrac{1}{2i}\left(e^{it}-e^{-it}\right) &= \sin t \end{aligned}}$$

Also, $\boxed{e^{i\pi} = -1}$ Visually,



$\theta = \pi$ radians.

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \frac{(it)^6}{6!} + \frac{(it)^7}{7!} + \frac{(it)^8}{8!} + \dots$$

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} +/- \dots$$

$$i\sin t = it - i\frac{t^3}{3!} + i\frac{t^5}{5!} - i\frac{t^7}{7!} +/- \dots$$

$\boxed{e^{it} = \cos t + i\sin t}$

Question: How do we solve $y''+4y'=0$

Is this "really" a 2^{nd} order ODE?

What if $\left.\begin{array}{l} v=y' \\ v'=y'' \end{array}\right\} \Rightarrow v'-4v=0$

$v(t)=Ce^{4t}=y'(t) \Rightarrow \boxed{y(t)=Ce^{4t}+D}$

Is this the general solution? (Yes!)

Question: How to solve $y''=-y-1$?

let $\left.\begin{array}{l} v=-y-1 \\ v'=-y' \end{array}\right\} \Rightarrow v''=-v \Rightarrow v(t)=A\cos t+B\sin t$

$y(t)=A\cos t+B\sin t-1.$

We can also make a change of variables to turn a 2^{nd} order ODE into a system of two 1^{st} order ODEs.

$\left.\begin{array}{l} y''=f(t,y,y') \\ \text{let } v=y' \\ \quad v'=y'' \end{array}\right\} \Longrightarrow \begin{array}{l} v'=f(t,y,y') \\ v=y' \end{array}$

Example: $y''=3y'+y+t^2 \rightsquigarrow \begin{cases} v'=3v+y^2+t^2 \\ y'=v \end{cases}$

Why do we do this?

- Many numerical methods (e.g., Euler's method) only work for 1^{st} order ODEs.

- Can deduce results about higher order systems by studying 1^{st} order systems.

Ex: Instead of $x(t)$, $x'(t)$, and $x''(t)$,

$$\text{use } x(t),\ v(t),\ \text{and } v'(t).$$

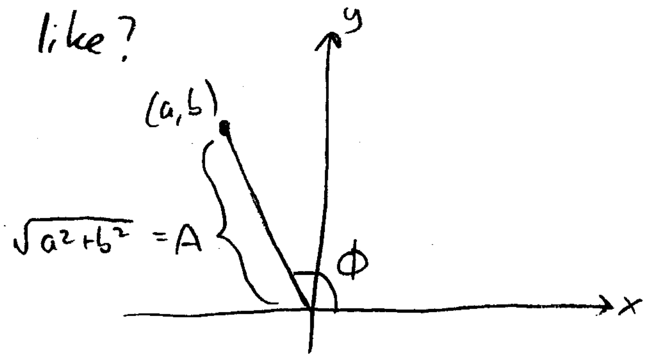
Harmonic Motion

First, think about what a function of the form

$x(t) = a\cos\omega_0 t + b\sin\omega_0 t$ looks like?

Let's switch to polar coordinates:

$(a,b) = (A\cos\phi,\ A\sin\phi)$



$A \geq 0, \quad -\pi < \phi \leq \pi$

Sneaky little trick:

$$x(t) = \underbrace{a}\cos(\omega_0 t) + \underbrace{b}\sin(\omega_0 t)$$

$$\left\{\begin{aligned} &= \underbrace{A\cos\phi}\cos(\omega_0 t) + \underbrace{A\sin\phi}\sin(\omega_0 t) \\ &= A\cos(\phi - \omega_0 t) = A\cos(\omega_0 t - \phi) \end{aligned}\right.$$

(Recall: $\cos(x-y) = \cos x\cos y + \sin x \sin y$)

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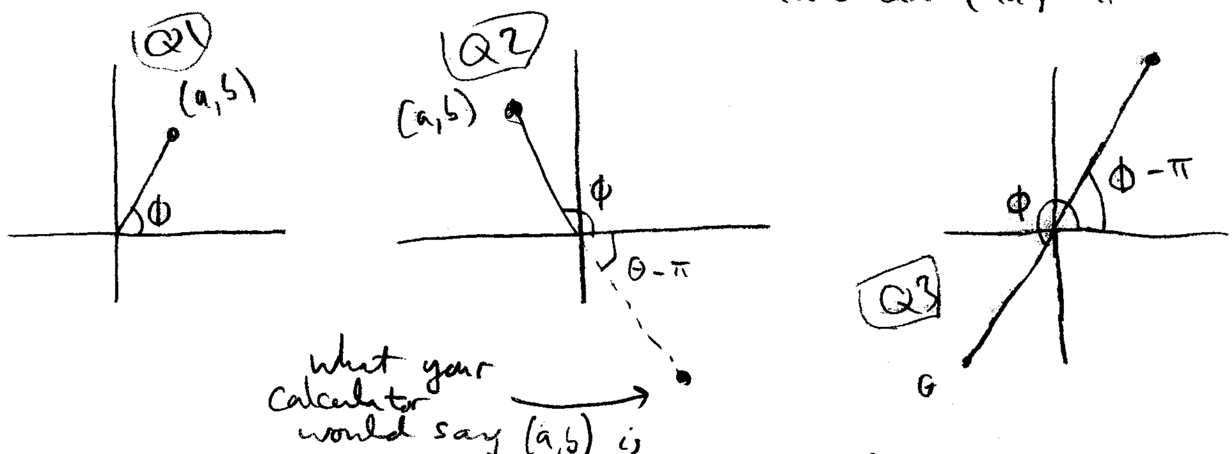
<u>Big idea</u>: Any function $x(t) = a\cos(\omega_0 t) + b\sin(\omega_0 t)$

can be written as a <u>single cosine wave</u>, with

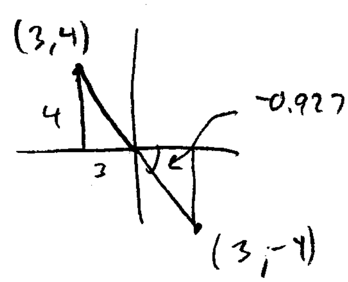
* amplitude $A = \sqrt{a^2+b^2}$
* phase shift $\frac{\phi}{\omega_0}$, where "$\phi = \tan^{-1}(b/a)$".

$$x(t) = A\cos(\omega_0 t - \phi) = \boxed{A\cos\left(\omega_0\left(t - \frac{\phi}{\omega_0}\right)\right)}$$

<u>Note</u>: since $\underbrace{-\frac{\pi}{2} < \phi \le \frac{\pi}{2}}_{\text{for calculators}}$, $\phi = \begin{cases} \arctan^{-1}(b/a) & Q1,4 \\ \arctan^{-1}(b/a) + \pi & Q2 \\ \arctan^{-1}(b/a) - \pi & Q3 \end{cases}$



<u>Example</u>: $x(t) = -3\cos t + 4\sin t$



$A = \sqrt{3^2+4^2} = 5$

$\arctan(-4/3) = -0.927$ ← what your calculator would say

so $\phi = -0.927 + \pi$

$\Rightarrow x(t) = 5\cos[t - (-0.927+\pi)]$