MTHSC 208, WEEK 6

Week 5 summary

- 2^{nd} order linear ODEs have solutions of the form

$$y(t) = y_h(t) + y_p(t) = C_1\, y_1(t) + C_2\, y_2(t) + y_p(t).$$

- 2^{nd} order linear homogeneous ODEs (with const. coefficients)

$$y'' + p y' + q y = 0.$$

Assume $y(t) = e^{rt}$, get characteristic polynomial $r^2 + pr + q = 0$.

3 cases:

(i)	$r_1 \neq r_2$ real	$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
(ii)	$r_1 = r_2$	$y(t) = C_1 e^{r_1 t} + C_2 t e^{r_1 t}$
(iii)	$r_{1,2} = a \pm bi$	$y(t) = e^{at}(A \cos bt + B \sin bt)$

- Euler's equation: $e^{i\theta} = \cos\theta + i \sin\theta$

$$\Rightarrow \cos\theta = e^{i\theta} + e^{-i\theta}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

- Harmonic motion (prereqs)

If $y(t) = a \cos(\omega_0 t) + b \sin(\omega_0 t)$, then $y(t) = A\cos(\omega_0 t - \phi)$, where $A = \sqrt{a^2+b^2}$, and "$\phi = \arctan(b/a)$"

This week:
- Harmonic motion
- Solving 2^{nd} order linear inhomog. ODEs.

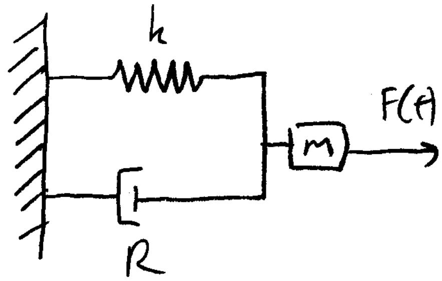
Harmonic motion:

Mass-spring systems

$y(t)$ = displacement

$$my'' + \mu y' + ky = F(t)$$

damping force — spring force — driving force

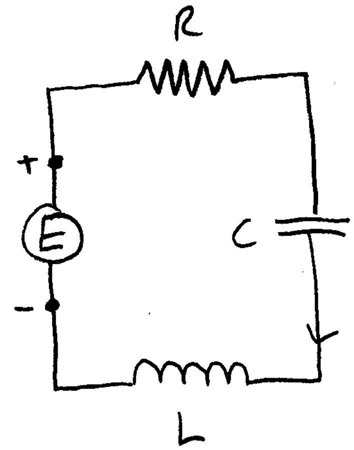


RLC circuits:

$I(t)$ = current

$$LI'' + RI' + \frac{1}{C}I = \frac{dE}{dt}$$

inductance — Resistance — capacitance — source voltage



Mathematically, these are the same!

Whichever setting we're in, we can simplify this ODE to

$$x'' + 2cx' + \omega_0^2 x = f(t) \qquad c \geq 0, \quad \omega_0 > 0.$$

This is the equation for harmonic motion.

- c is the damping constant
- $f(t)$ is the forcing term.

In this section, we will assume $f(t) = 0$.

First case: $c=0$ (simple harmonic motion)

$$x'' + \omega_0^2 x = 0$$ (Remember: sines & cosines!)

$$x(t) = a\cos\omega_0 t + b\sin\omega_0 t$$

period

(Hence why we call ω_0 "frequency": $x(t) = x(t + \frac{2\pi}{\omega_0})$)

Next case: $c \neq 0$ (damped harmonic motion)

$$x'' + 2cx' + \omega_0^2 x = 0 \qquad c>0$$

Assume $x(t) = e^{rt} \implies r^2 + 2cr + \omega_0^2 = 0$

$$\implies r = -c \pm \sqrt{c^2 - \omega_0^2}$$

3 cases:

(i) Complex roots $(c<\omega_0)$: underdamped

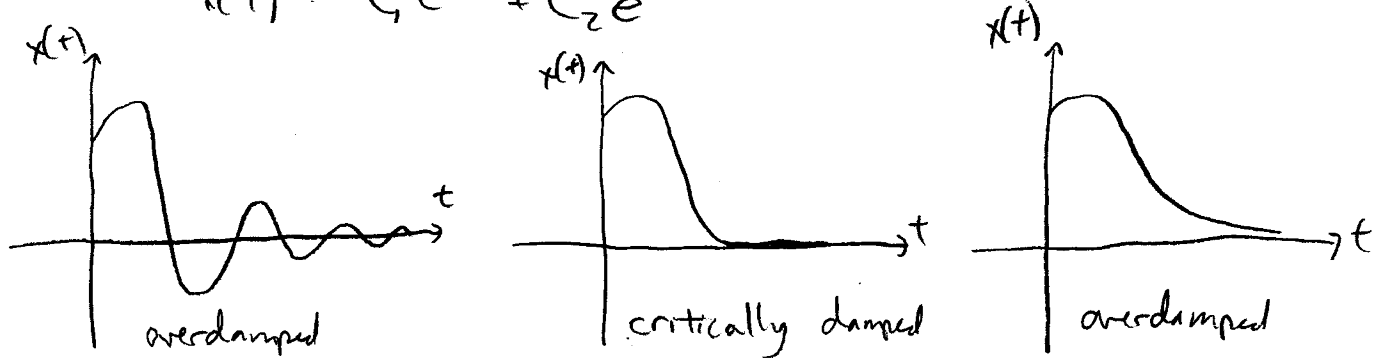
$$x(t) = e^{-ct}(a\cos\omega_0 t + b\sin\omega_0 t)$$

(ii) Double root $(c=0)$ critically damped

$$x(t) = C_1 e^{r_1 t} + C_2 t e^{r_1 t} \quad (\text{note: } r_1 = r_2 < 0)$$

(iii) 2 real roots $(c>\omega_0)$ overdamped

$$x(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t}$$



Inhomogeneous linear 2nd order ODEs:

$$y'' + py' + qy = f \qquad p(t),\ q(t),\ f(t).$$

* Big idea: $y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$, where y_1, y_2 are solutions to the homogeneous equation, and $y_p(t)$ is any solution to the inhomogeneous eq'n.

Proof: Suppose we have a particular sol'n $y_p(t)$.

Let $y(t)$ be any other solution. Then

$$\begin{array}{l} y'' + py' + qy = F \\ -(y_p'' + py_p' + qy_p = F) \\ \hline (y-y_p)'' + p\cdot(y-y_p)' + q\cdot(y-y_p) = 0 \end{array}$$

Thus, $y - y_p$ is a solution to the homogeneous eq'n.

i.e., $y - y_p = C_1 y_1 + C_2 y_2 \Rightarrow y = C_1 y_1 + C_2 y_2 + y_p$.

Method of undetermined coefficients

- Works only when coefficients p, q are constants.

* Big idea: Usually we can guess the form of a particular sol'n.

Example: $y'' - 5y' + 4y = e^{3t}$

First, solve the homog. eq'n: $y_h(t) = C_1 e^{4t} + C_2 e^{t}$

Next, guess that $y_p(t) = a e^{3t}$ (why will this work??)

$$y_p'(t) = 3ae^{3t}, \quad y_p''(t) = 9ae^{3t}$$

Plug back in & solve for a.

$$9ae^{3t} - 5(3ae^{3t}) + 4ae^{3t} = e^{3t}$$

$$-2ae^{3t} = e^{3t} \Rightarrow a = -\tfrac{1}{2}$$

Thus, $y_p(t) = -\frac{1}{2}e^{3t}$, and $y(t) = y_h(t) + y_p(t)$

$$\Rightarrow \boxed{y(t) = C_1 e^{4t} + C_2 e^{t} - \tfrac{1}{2}e^{3t}}$$

Why did this work?

* Because the forcing term $f(t)$ and the derivative had the same form (up to constant).

What if the forcing term is $f(t) = \sin t$?

Example: $y'' + 2y' - 3y = 5\sin 3t$?

Problem: $f(t) = 5\sin 3t$, $f'(t) = 15\cos 3t$.

Not of the same form. How do we fix this?

Ans: Consider a more general forcing term.

Observe: $\left.\begin{array}{l} f(t) = A\cos wt + B\sin wt \\ f'(t) = -wA\cos wt + wB\sin wt \end{array}\right\}$ have the same form!

$\boxed{6}$

Assume there's a particular solution of the form

$$\boxed{y_p(t) = a \cos 3t + b \sin 3t}$$

$$y_p'(t) = -3a \sin 3t + 3b \cos 3t$$

$$y_p''(t) = -9a \cos 3t - 9b \sin 3t$$

$$y_p'' + 2y_p' - 3y_p = (-9a \cos 3t - 9b \sin 3t) + (-6a \sin 3t + 6b \cos 3t)$$
$$-(3a \cos 3t + 3b \sin 3t)$$
$$= \underbrace{(-12a + 6b)}_{=0} \cos 3t + \underbrace{(-6a - 12b)}_{=5} \sin 3t = \underbrace{5 \sin 3t}_{f(t)}$$

Thus, we have $\left.\begin{array}{l} -12a + 6b = 0 \\ -6a - 12b = 5 \end{array}\right\} \Rightarrow a = -\frac{1}{6}, \quad b = -\frac{1}{3}$

$$\Rightarrow y_p(t) = \frac{-1}{6} \cos 3t - \frac{1}{3} \sin 3t.$$

The general solution is therefore

$$y(t) = y_h(t) + y_p(t) = \boxed{C_1 e^t + C_2 e^{-3t} + \left(-\frac{1}{6} \cos 3t - \frac{1}{3} \sin 3t\right)}$$

<u>Example:</u> (polynomial forcing term)

$y'' + 2y' - 3y = 3t + 4$. Again, $y_h(t) = C_1 e^t + C_2 e^{-3t}$

Assume there's a particular solution of the form $\boxed{y_p(t) = at + b}$

Why? B/c $y_p'' + 2y_p' - 3y_p$ will also be be a 1st degree poly.

So all we have to do is find a & b.

Plug back in $(y_p' = a, \ y_p'' = 0)$

$$y_p'' + 2y_p' - 3y_p = 0 + 2a - 3(at+b) = \underbrace{-3a}_{=3}\,t + \underbrace{2a-3b}_{=4} = 3t+4$$

We get $\left.\begin{array}{l} -3a = 3 \\ 2a-3b=4 \end{array}\right\} \Rightarrow a=-1,\ b=2 \Rightarrow y_p(t) = -t-2$

Thus, the general solution is

$$y(t) = y_h(t) + y_p(t) = \boxed{C_1 e^{t} + C_2 e^{-3t} - (t+2)}$$

- What could go wrong with this method?

What if the forcing term is a solution to the homogeneous eqn?

ex: $y'' - y' - 2y = 3e^{-t}$

char. eq'n: $e^{rt}(r-2)(r+1) = 0 \Rightarrow y_h(t) = C_1 e^{2t} + C_2 e^{-t}$.

Will there be a particular solution of the form $y_p(t) = ae^{-t}$?

No! Because if we plug it back in, we get $0 = 3e^{-t}$.

So, assume there's a solution of the form $\boxed{y_p(t) = ate^{-t}}$

$y_p'(t) = a(1-t)e^{-t}$, $y_p''(t) = a(t-2)e^{-t}$, and

$$y_p'' - y_p' - 2y_p = \underbrace{a(t-2)e^{-t} - a(1-t)e^{-t} - 2ate^{-t}}_{=-3ae^{-t}} = \underbrace{3e^{-t}}_{f(t)}$$

Thus $-3ae^{-t} = 3e^{-t} \Rightarrow a = -1 \Rightarrow y_p(t) = -te^{-t}$

and $y(t) = y_h(t) + y_p(t) = \boxed{C_1 e^{2t} + C_2 e^{-t} - te^{-t}}$

Combination forcing terms:

Suppose $y'' + py' + qy = f(t)$ has sol'n $y_F(t)$

and $y'' + py' + qy = g(t)$ has sol'n $y_g(t)$.

Then $y'' + py' + qy = \alpha f(t) + \beta g(t)$ has sol'n $\alpha y_F(t) + \beta y_g(t)$.

Application: Consider $y'' + 2y' - 3y = 5 \sin 3t + 3t + 4$

We found $y_p(t) = -\frac{1}{6} \cos 3t - \frac{1}{3} \sin 3t$ for $y'' - 2y' - 3y = 5 \sin 3y$

and $y_p(t) = -(t+2)$ for $y'' - 2y' - 3y = 3t + 4$.

Thus, we get $y_p(t) = -\frac{1}{6} \cos 3t - \frac{1}{3} \sin 3t - (t+2)$

and a general solution $\boxed{y(t) = C_1 e^{t} + C_2 e^{-3t} - \frac{1}{6} \cos 3t - \frac{1}{3} \sin 3t - (t+2)}$

Forced harmonic motion:

- spring attached to a motor
- source voltage is sinusoidal

$$x'' + 2cx' + \omega_0^2 x = A \cos \omega t$$

damping coeffic.

natural freq.

driving frequency

Sample case: No damping $(c = 0)$.

$$x'' + \omega_0^2 x = A \cos \omega t$$

Homog. eq'n: $x_h'' + \omega_0^2 x = 0$

$$x_h = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

Case 1: $\omega \neq \omega_0$

$x_p = a \cos \omega t + b \sin \omega t$. Need to solve for a, b.

plug x_p back in:

$$x_p'' + \omega_0^2 x_p = a(\omega_0^2 - \omega^2)\cos \omega t + b(\omega_0^2 - \omega^2)\sin \omega t$$
$$= A \cos \omega t + 0 \sin \omega t$$

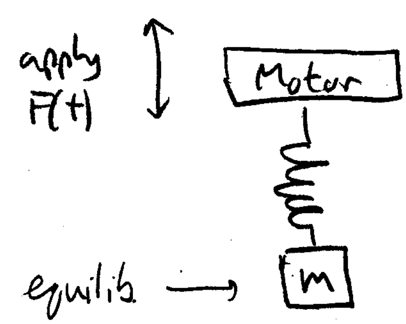
$$\Rightarrow \left.\begin{array}{l} a(\omega_0^2 - \omega^2) = A \\ b(\omega_0^2 - \omega^2) = 0 \end{array}\right\} \Rightarrow a = \frac{A}{\omega_0^2 - \omega^2}, \quad b = 0$$

i.e., $x_p(t) = \dfrac{A}{\omega_0^2 - \omega^2} \cos \omega t$

General solution: $x(t) = x_h(t) + x_p(t)$

$$= \boxed{C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{A}{\omega_0^2 - \omega^2} \cos \omega t}$$

What does this solution look like?



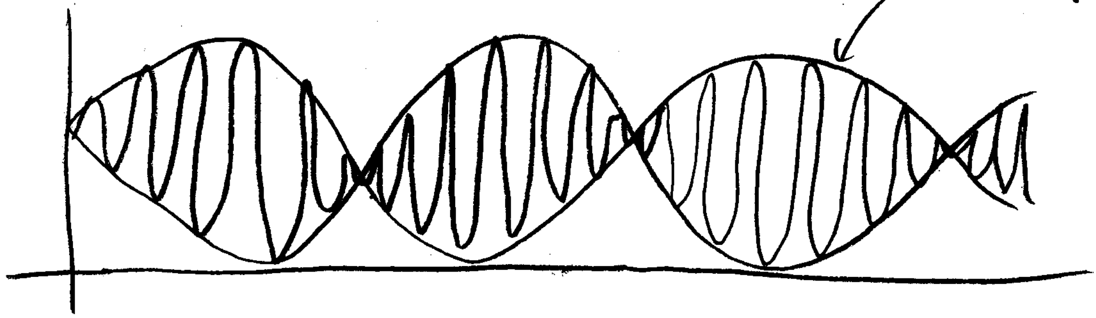
First, consider equilibrium: $x(0) = 0$
$x'(0) = 0$

$c_2 = 0$, $c_1 = \dfrac{-A}{\omega_0^2 - \omega^2} \Rightarrow x(t) = \dfrac{A}{\omega_0^2 - \omega^2}(\cos \omega t - \cos \omega_0 t)$

- Superposition of waves with different frequencies

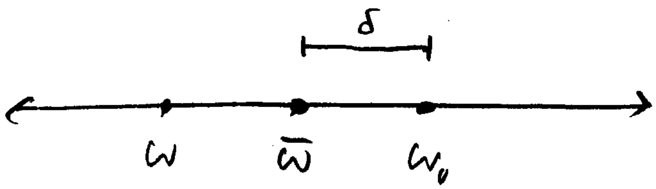
Has anyone experienced this in real life (think music!)

Example: $x(t) = \cos(11t) - \cos(12t)$



How to quantify this?

$$\omega = \bar{\omega} - \delta$$
$$\omega_0 = \bar{\omega} + \delta$$



$$\Rightarrow x(t) = \frac{A}{\omega_0^2 - \omega^2}\left(\cos \omega t - \cos \omega_0 t\right)$$

$$x(t) = \underbrace{\left(\frac{A \sin(\delta t)}{2\bar{\omega}\delta}\right)}_{\text{Amplitude is sinusoidal}} \sin \bar{\omega} t$$

Case 2: $\omega = \omega_0$ (Recall: $f(t) = A\cos \omega t$ is forcing term).

If $\omega = \omega_0$, then $f(t) = A \cos \omega t$ solves the homog. eq'n

Thus $x_p(t) = t(a \cos \omega_0 t + b \sin \omega_0 t)$

Plug x_p back in:

$$x_p'' + \omega_0^2 x_p = \left[2\omega_0(-a \sin \omega_0 t + b \cos \omega_0 t) + t\omega_0^2(-a \cos \omega_0 t + b \sin \omega_0 t)\right]$$
$$+ \omega_0^2 t(a \cos \omega_0 t + b \sin \omega_0 t)$$
$$= 2\omega_0(-a \sin \omega_0 t + b \cos \omega_0 t) = A \cos \omega t$$

$$\left.\begin{aligned} -2\omega_0 a &= 0 \\ 2\omega_0 b &= A \end{aligned}\right\} \Rightarrow a = 0, \quad b = \frac{A}{2\omega_0}$$

Thus $x_p(t) = \dfrac{A}{2\omega_0} t \sin \omega_0 t$

General sol'n $x(t) = x_h(t) + x_p(t)$

$$= \boxed{C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \overbrace{\frac{A}{2\omega_0} t}^{\text{Amplitude} \to \infty!} \sin \omega_0 t}$$

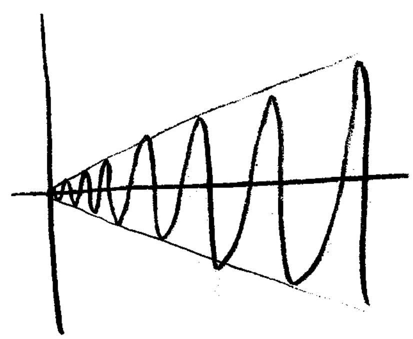
Look at the long-term behavior. This wave "blows up"!

Example Again, consider equilibrium: $x(0) = 0$, $x'(0) = 0$

$x(0) = C_1 = 0$

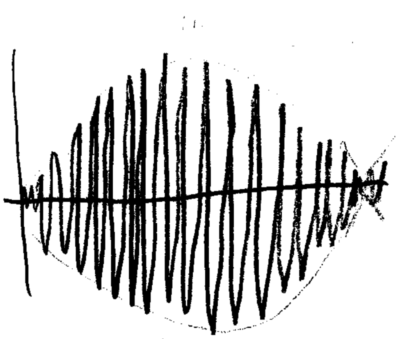
$x'(t) = C_2 \omega_0 \cos \omega_0 t + \frac{A}{2} t \cos \omega_0 t + \frac{A}{2\omega_0} \sin \omega_0 t$

$x'(0) = C_2 = 0 \Rightarrow x(t) = \dfrac{A}{2\omega_0} t \sin \omega_0 t$

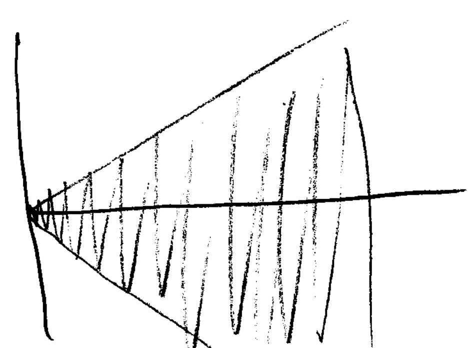


Real-life application: Tacoma Narrows Bridge

Compare $\omega_0 \approx \omega$ to $\omega_0 = \omega$



$\omega_0 \approx \omega$: envelope "closes up"



$\omega_0 = \omega$: envelope never closes up (like dividing by zero)