MTHSC 208, WEEK 7

Week 6 summary

- Method of undetermined coefficients

Used to solve linear inhomogeneous 2nd order ODE's with constant coefficients.

Big idea: Guess $y_p(t)$ to have the same form as the forcing term $f(t)$. Then $y(t) = y_h(t) + y_p(t)$

$f(t)$	$y_p(t)$
e^{rt}	ae^{rt}
$\cos wt$ or $\sin wt$	$a\cos wt + b\sin wt$
n^{th} deg polynomial	n^{th} degree polynomial
$e^{rt}\cos wt$ or $e^{rt}\sin wt$	$e^{rt}(a\cos wt + b\sin wt)$
linear combin. of above f^ns	linear combin. of above f^ns

- Harmonic motion: $x'' + 2cx' + w_0^2 x = f(t)$.

* Simple harmonic motion $(c = f(t) = 0)$:

$x(t) = a\cos w_0 t + b\sin w_0 t$ or $A\cos(w_0 t - \phi)$.

* With damping $(c \neq 0)$. Roots $r_{1,2} = -c \pm \sqrt{c^2 - w_0^2}$

case 1: $c > w_0$ overdamped $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

case 2: $c = w_0$ critically damped $x(t) = C_1 e^{rt} + C_2 t e^{rt}$

case 3: $c < w_0$ underdamped $x(t) = e^{-ct}(a\cos w_0 t + b\sin w_0 t)$

* Forced harmonic motion ($f(t) \neq 0$).

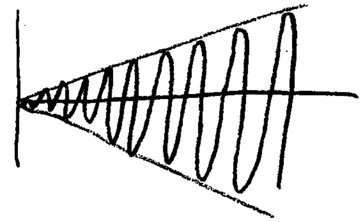
example IVP

e.g., $x'' + w_0^2 x = A \cos wt$

Case 1: $w \neq w_0$. $x(t) = \frac{A}{w_0^2 - w^2}(\cos wt - \cos w_0 t) = \frac{A}{2\delta\bar{w}} \sin \delta t \sin \bar{w} t$

Case 2: $w = w_0$ $x(t) = C_1 \cos w_0 t + C_2 \sin w_0 t + \frac{A}{2w_0} t \sin w_0 t$

Example IVP



<u>This week</u>: Laplace transforms

- used to solve linear ODE's
- Useful when forcing term is discontinuous

  e.g, step function

Think: Force being turned on/off.

Big idea:

$$\begin{tikzcd} \text{ODE} \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & \text{Algebraic eq'n} \arrow[d] \\ \text{sol'n to ODE} & \text{sol'n to alg eq'n} \arrow[l, "\mathcal{L}^{-1}"'] \end{tikzcd}$$

The Laplace transform is an operator: inputs a function, spits outs a function.

<u>Def:</u> Suppose $f(t)$ is defined for $0 < t < \infty$. The <u>Laplace transform</u> of f is the function $\mathcal{L}(f)$, where

$$\boxed{\mathcal{L}(f)(s) = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \qquad s > 0}$$

Often, we denote $\mathcal{L}(f)$ by F, i.e, $f \longmapsto F$.

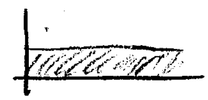
Recall: $\int_0^\infty f(t)\,dt := \lim_{T\to\infty} \int_0^T f(t)\,dt$.

<u>Example:</u> Compute $\mathcal{L}(f)$, where $f(t) = e^{at}$.

$$F(s) = \int_0^\infty e^{at} e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt.$$

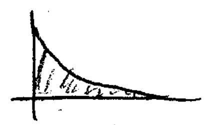
3 <u>cases:</u>

(i) $s=a$: $F(s) = \int_0^\infty 1\,dt = \infty$ (undefined)



(ii) $s<a$: $F(s) = \int_0^\infty e^{-(s-a)t}\,dt = \int_0^\infty e^{|s-a|t}\,dt = \infty$ (undefined)

(iii) $s>a$: $F(s) = \int_0^\infty e^{-(s-a)t}\,dt = \lim_{T\to\infty} \int_0^T e^{-(s-a)t}$



$$= \lim_{T\to\infty} \left. \frac{e^{-(s-a)T}}{-(s-a)} \right|_0^\infty = \lim_{T\to\infty} \left(-\frac{e^{-(s-a)T}}{s-a} + \frac{1}{s-a} \right) = \frac{1}{s-a}$$

Thus, $\mathcal{L}(e^{at})(s) = F(s) = \frac{1}{s-a}$ for $s>a$.

<u>Note:</u> Sometimes the domain is restricted.

e.g., f has domain $(-\infty, \infty)$

F has domain (a, ∞)

<u>Recall:</u> Integration by parts

Let's rederive it: $(uv)' = u\,dv + du\,v$

$$u\,dv = (uv)' - du\,v$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example: Let $F(t) = t$. Compute $\mathcal{L}(F)$

$$F(s) = \int_0^\infty t e^{-st}\, dt \qquad \begin{aligned} &\text{let } u = t \\ &du = dt \end{aligned} \qquad \begin{aligned} &v = -\tfrac{1}{s} e^{-st} \\ &\text{let } dv = e^{-st}\, dt \end{aligned}$$

$$\int \underbrace{t}_{u}\, \underbrace{e^{-st} dt}_{dv} = \underbrace{-\tfrac{1}{s} t e^{-st}}_{uv} + \underbrace{\tfrac{1}{s} \int e^{-st}\, dt}_{-\int v\, du} = \frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2}$$

$$\mathcal{L}(t) = F(s) = \int_0^\infty t e^{-st}\, dt = \lim_{T\to\infty} \int_0^T t e^{-st}\, dt$$

$$= \lim_{T\to\infty} \left(\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2} \right) \Bigg|_0^T = \lim_{T\to\infty} \left(\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2} \right) - \left(0 - \frac{e^0}{s^2} \right) = \frac{1}{s^2}$$

Other common functions:

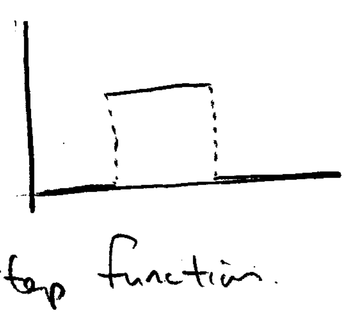
$$\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(\sin at)(s) = \frac{a}{s^2+a^2}$$

$$\mathcal{L}(\cos at)(s) = \frac{s}{s^2+a^2}$$

We can also compute the Laplace transform of piecewise continuous & piecewise differentiable functions.

e.g.,



step function
piecewise continuous

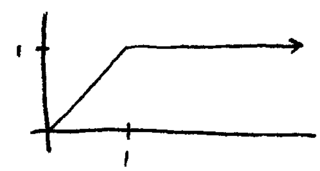
continuous
piecewise differentiable.

Ex: let $f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$

Compute $\mathcal{L}(F)(s) = F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}\, dt$

$$= -\frac{1}{s} e^{-st} \Big|_0^1 = \boxed{\frac{-e^{-s}}{s} + \frac{1}{s}}$$

Ex: let $f(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 & 1 < t < \infty \end{cases}$



We must break the integral into 2 parts

$$\mathcal{L}(F)(s) = F(s) = \int_0^\infty e^{-st} f(t)\, dt = \underbrace{\int_0^1 t e^{-st}\, dt}_{I1} + \underbrace{\int_1^\infty e^{-st}\, dt}_{I2}$$

$$I1 = -\frac{e^{-s}}{s} - \left(\frac{e^{-s}}{s^2} - \frac{1}{s^2}\right) \qquad I2 = \lim_{T \to \infty} \frac{-e^{-st}}{s}\Big|_{t=1}^{T} = \frac{e^{-s}}{s}$$

$$F(s) = \left(-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right) + \left(\frac{e^{-s}}{s}\right) = \boxed{\frac{1}{s^2} - \frac{e^{-s}}{s^2}}$$

Properties of the Laplace transform:

- $\mathcal{L}$ is linear
- $\mathcal{L}$ turns derivatives into multiplication.

(and we'll see a few others as well).

(i) Linearity: $\mathcal{L}(a f(t) + b g(t))(s) = a \mathcal{L}(f(t)) + b \mathcal{L}(f(t))$

i.e., you can break apart sums & pull apart constants.

($\mathcal{L}$ is a "linear operator").

<u>ex:</u> $\mathcal{L}(6t^4 - 2\sin 3t + 4\cos 5t)$

$$= 5\mathcal{L}(t^4) - 2\mathcal{L}(\sin 3t) + 4\mathcal{L}(\cos 5t)$$

$$= 6 \cdot \frac{4!}{s^5} - 2 \cdot \frac{3}{s^2+9} - 4\,\frac{s}{s^2+25}$$

(ii) <u>Turns derivatives into multiplication</u>

$$\mathcal{L}(y'(t))(s) = sY(s) - y(0)$$

<u>Proof:</u> $\mathcal{L}(y')(s) = \int_0^\infty y'(t)e^{-st}\,dt = \lim_{T\to\infty}\Big[e^{-st}y(t) + \overbrace{s\int_0^T y(t)e^{-st}\,dt}^{s\mathcal{L}(y)(s)}\Big]$

$$= \lim_{T\to\infty} e^{-st}y(t)\Big|_0^T + s\mathcal{L}(y)(s)$$

$$= \underbrace{\lim_{T\to\infty} e^{-sT}y(t)}_{\to 0 \text{ as long as } |y(t)| \le Ce^{at}} - y(0) + s\mathcal{L}(y)(s) = \boxed{s\mathcal{L}(y)(s) - y(0)}$$

Similarly, $\mathcal{L}(y'')(s) = s^2Y(s) - sy(0) - y'(0)$

$$\mathcal{L}(y''')(s) = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

$$\mathcal{L}(y^{(4)})(s) = s^4Y(s) - s^3y(0) - s^2y''(0) - sy'''(0)$$

$\vdots$

etc.

<u>Proof:</u> $\mathcal{L}(y'') = s\mathcal{L}(y')(s) - y'(0)$

$$= s(sY(s) - y(0)) - y'(0) = s^2Y(s) - sy(0) - y'(0)$$

and so on, inductively. $\square$

<u>Application:</u> Consider $y'' - y = e^{2t}$ $\quad y(0) = 0, \; y'(0) = 1.$

$$\mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}(e^{2t})$$

$$\left[s^2 Y(s) - s\cancelto{0}{y(0)} - \cancelto{1}{y'(0)}\right] - \left[Y(s)\right] = \frac{1}{s-2}$$

$$s^2 Y - 1 - Y = \frac{1}{s-2} \implies (s^2 - 1)Y = \frac{1}{s-2} + \frac{s-2}{s-2} = \frac{s-1}{s-2}$$

$$\boxed{Y(s) = \frac{1}{(s+1)(s-2)}}$$

* The solution to the IVP (above) is the function whose Laplace transform is $Y(s) = \frac{1}{(s+1)(s-2)}$.
(we'll show how to do this later).

$$\begin{tikzcd}
y'' - y = e^{2t} \quad \begin{matrix} y(0)=0 \\ y'(0)=1 \end{matrix} \arrow[rr, "\mathcal{L}"] \arrow[d, dashed] & & s^2 Y - 1 - Y = \frac{1}{s-2} \arrow[d, "\text{solve!}"] \\
\text{Solution } y(t) & & Y(s) = \frac{1}{(s+1)(s-2)} \arrow[ll, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

<u>More Laplace transform facts</u>

(i) $\mathcal{L}\{e^{ct} f(t)\}(s) = F(s-c)$

(ii) $\mathcal{L}\{t\, f(t)\}(s) = -F'(s)$

(iii) $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s)$

Applications of this:

Ex 1: $f(t) = e^{2t}\cos 3t$

Recall: $\mathcal{L}(\cos 3t) = \frac{s}{s^2+9}$.

Using (i), $\mathcal{L}(e^{2t}\cos 3t) = \frac{(s-2)}{(s-2)^2+9} = F(s)$

Ex 2: $f(t) = t^2 e^{3t}$. let $g(t) = e^{3t}$

Recall: $\mathcal{L}(e^{3t}) = \frac{1}{s-3} = G(s)$.

Using (iii), $\mathcal{L}(t^2 e^{3t}|(s) = (-1)^2 F''(s) = 1\ \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right) = \frac{2}{(s-3)^3}$

Back to using Laplace transforms to solve ODEs.

Example: $y' - 4y = \cos 2t \qquad y(0) = -2$

$\mathcal{L}(y') - \mathcal{L}(4y) = \mathcal{L}(\cos 2t)$

$[sY - y(0)] - 4Y = \frac{s}{s^2+4}$

$sY + 2 - 4Y = \frac{s}{s^2+4} \implies (s-4)Y = \frac{s}{s^2+4} - \frac{2(s^2+4)}{s^2+4} = \frac{-2s^2+s-8}{s^2+4}$

$\implies Y(s) = \frac{-2s^2+s-8}{(s-4)(s^2+4)}$

Example: $y'' + 3y' + 5y = t + e^{-t} \qquad y(0) = -1, \quad y'(0) = 0.$

$\mathcal{L}(y'') + 3\mathcal{L}(y') + 5\mathcal{L}(y) = \mathcal{L}(t) + \mathcal{L}(e^{-t})$

$[s^2Y - s\,y(0) - y'(0)] + 3[sY - y(0)] + 5Y = \frac{1}{s^2} + \frac{1}{s^2+1}$

$s^2Y + s + 3sY + 3 + 5Y = \frac{s^2+1}{s^2(s^2+1)} + \frac{s^2}{s^2(s^2+1)}$

$$(s^2+3s+5)Y + (s+3) = \frac{2s^2+1}{s^2(s^2+1)}$$

$$(s^2+3s+5)Y = \frac{2s^2+1}{s^2(s^2+1)} - \frac{s^2(s^2+1)(s+3)}{s^2(s^2+1)}$$

$$(s^2+3s+5)Y = \frac{(2s^2+1)-(s^5+3s^4+s^3+3s^2)}{s^2(s^2+1)}$$

$$\Rightarrow Y(s) = \frac{-s^5-3s^4-s^3-s^2+1}{s^2(s^2+1)(s^2+3s+5)}$$

Let's revisit the IVP $y''-y=e^{2t}$, $y(0)=0$, $y'(0)=1$.

Recall that $Y(s) = \frac{1}{(s+1)(s-2)}$.

To solve for $y(t)$, we must compute $\mathcal{L}^{-1}\left(\frac{1}{(s+1)(s-2)}\right)$.

To do this, write $\frac{1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$.

Use partial fraction decomposition.

$$\frac{A(s-2)}{(s+1)(s-2)} + \frac{B(s+1)}{(s-2)(s+1)} = \frac{1}{(s+1)(s-2)} \Rightarrow \underbrace{(A+B)}_{=0}s + \underbrace{(B-2A)}_{=1} = 1$$

$$\Rightarrow \begin{cases} A+B=0 \\ B-2A=1 \end{cases}$$

$A=-B \Rightarrow 3B=1 \Rightarrow B=1/3$, $A=-1/3$

So, $\frac{1}{(s+1)(s-2)} = \frac{-1/3}{s+1} + \frac{1/3}{s-2}$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{(s+1)(s-2)}\right) = \mathcal{L}^{-1}\left(\frac{-1/3}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{1/3}{s-2}\right)$$

$$= -\frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{3}\mathcal{L}\left(\frac{1}{s-2}\right) = \boxed{-\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}}$$

Example: Compute $\mathcal{L}^{-1}\left(\frac{1}{s^2+4s+13}\right)$

Goal: Put it in the form $\frac{1}{(s-b)^2+a^2}$, because

$$\mathcal{L}(e^{bt}\cos at) = \frac{a}{(s-b)^2+a^2}$$

$$\frac{1}{(s^2+4s+4)+9} = \frac{1}{(s+2)^2+3^2} = \frac{1}{3}\,\frac{3}{(s+2)^2+3^2}.$$

Example: Solve the IVP $y''-2y'-3y=0$, $y(0)=1$, $y'(0)=0$

Old method: $y(t)=e^{rt}$, $e^{rt}(r^2-2r-3)=0$

$$\Rightarrow (r-3)(r+1)=0$$

$$\Rightarrow y(t) = C_1 e^{3t} + C_2 e^{-t}$$

$y(0) = C_1 + C_2 = 1$ and $y'(t) = 3C_1 e^{3t} - C_2 e^{-t}$

$$y'(0) = 3C_1 - C_2 = 0$$

$$\Rightarrow \begin{cases} C_1 + C_2 = 1 \\ 3C_1 - C_2 = 0 \end{cases} \Rightarrow C_1 = \frac{1}{4},\ C_2 = \frac{3}{4} \Rightarrow \boxed{y(t) = \frac{1}{4}e^{3t} + \frac{3}{4}e^{-t}}$$

New method: $y''-2y'-3y=0$, $y(0)=1$, $y'(0)=0$

$$\mathcal{L}(y'') - 2\mathcal{L}(y') - 3\mathcal{L}(y) = 0$$

$$[s^2Y - sy(0) - y'(0)] - 2[sY - y(0)] - 3Y = 0.$$

$$[s^2Y - s - 0] - 2[sY - 1] - 3Y = 0$$

$$(s^2-2s-3)Y = s-2$$

$$Y = \frac{s-2}{s^2-2s-3} = \frac{A}{s-3} + \frac{B}{s+1} = \frac{s-2}{s^2-2s-3}$$

$$\frac{A}{s-3}\cdot\frac{s+1}{s+1} + \frac{B}{s+1}\frac{(s-3)}{(s-3)} = \frac{(A+B)s + (A-3B)}{(s+1)(s-3)} = \frac{s-2}{(s+1)(s-3)}$$

$$\begin{cases} A+B=1 \\ A-3B=-2 \end{cases} \Rightarrow \begin{matrix} A=\frac{1}{4} \\ B=\frac{3}{4} \end{matrix} \Rightarrow Y(s) = \frac{1/4}{s-3} + \frac{3/4}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1/4}{s-3}\right) + \mathcal{L}^{-1}\left(\frac{3/4}{s+1}\right) = \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) + \frac{3}{4}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$\boxed{y(t) = \frac{1}{4}e^{3t} + \frac{3}{4}e^{-t}}$$

Summary: Consider $ay''+by'+cy = F(t)$, $y(0)=y_0$, $y'(0)=y_1$

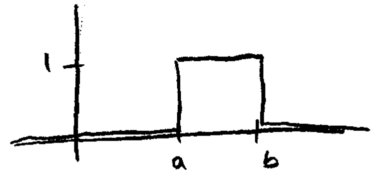
$$\begin{aligned}\mathcal{L}(ay''+by'+cy) &= a\mathcal{L}(y'')+b\mathcal{L}(y')+c\mathcal{L}(y) \\ &= a(s^2Y - sy(0) - y'(0)) + b(sY - y(0)) + cY \\ &= (as^2+bs+c)Y - y_0(as+b) - ay_1 \\ &= F(s).\end{aligned}$$

Thus $$Y(s) = \underbrace{\frac{F(s)}{as^2+bs+c}}_{\text{state-free sol'n}} + \underbrace{\frac{y_0(as+b)+ay_1}{as^2+bs+c}}_{\text{input-free sol'n}}$$

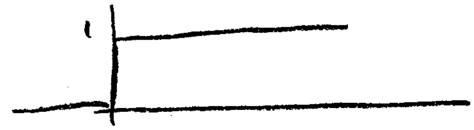
$$Y(s) = Y_s(s) + Y_i(s).$$

Discontinuous forcing terms

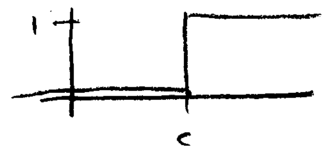
- Interval Function: $H_{ab}(t) = \begin{cases} 0 & t < a \\ 1 & a \le t < b \\ 0 & a \le t < \infty \end{cases}$



- Heavyside function: $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$

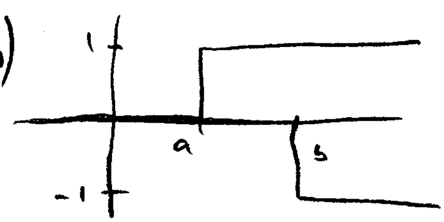


- Shifted Heavyside Function: $H_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$



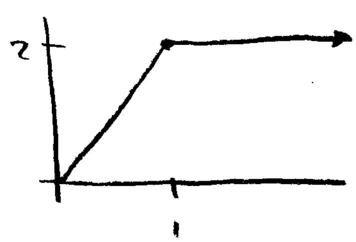
$$= H(t-c)$$

Note: $H_{ab}(t) = H_a(t) - H_b(t) = H(t-a) - H(t-b)$



* Many piecewise continuous functions can be represented using Heavyside functions.

Ex:



$$f(t) = \begin{cases} 2t & 0 \le t \le 1 \\ 2 & t \geq 1 \end{cases}$$

$$\begin{aligned} f(t) &= 2t\, H_{01}(t) + 2H_1(t) \\ &= 2t\,[H(t) - H(t-1)] + 2H(t-1) \\ &= 2t\,H(t) - 2(t-1)H(t-1). \end{aligned}$$