MTHSC 208, WEEK 8

Week 7 summary

- Laplace transforms: $\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad s>0$

$$\begin{tikzcd}
\text{ODE} \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & \text{Algebraic eq'n} \arrow[d] \\
\text{Sol'n to ODE} & \text{Sol'n to alg. eq'n} \arrow[l, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

  * Useful <u>especially</u> when forcing term $f(t)$ is discontinuous
  * $\mathcal{L}$ is linear
  * $\mathcal{L}$ turns derivatives into multiplication by s.

    e.g., $\mathcal{L}(y') = sY(s) - y(0)$

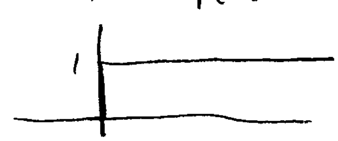
    $\mathcal{L}(y'') = s^2Y(s) - sy(0) - y'(0)$

- <u>Inverse Laplace transform</u>
  * Case 1: real roots e.g., $\frac{1}{(s-3)(s+1)}$ (Partial fractions)
  * Case 2: Complex roots e.g., $\frac{1}{s^2+4s+13}$

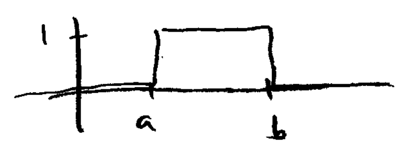
    (use table & complete the square)

- For an IVP w/ sol'n $y(t)$, $Y(s) = Y_s(s) + Y_i(s)$.

- Heavyside function: $H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t<0 \end{cases}$

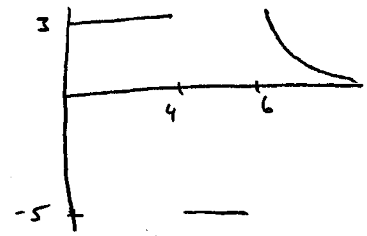


Interval function $H_{ab}(t) = H(t-a) - H(t-b)$



<u>Goal</u>: Use the Heavyside function to write discontinuous functions, so we can take their Laplace transform easily.

<u>Example</u>: $f(t) = \begin{cases} 3 & 0 \le t < 4 \\ -5 & 4 \le t < 6 \\ e^{7-t} & 6 \le t < \infty \end{cases}$



$$f(t) = 3H_{04}(t) - 5H_{46}(t) + e^{7-t}H_6(t)$$
$$= 3[H(t) - H(t-4)] - 5[H(t-4) - H(t-6)] + e^{7-t}[H(t-6)]$$
$$= 3H(t) - 8H(t-4) + 5H(t-6) + e^{7-t}H(t-6).$$

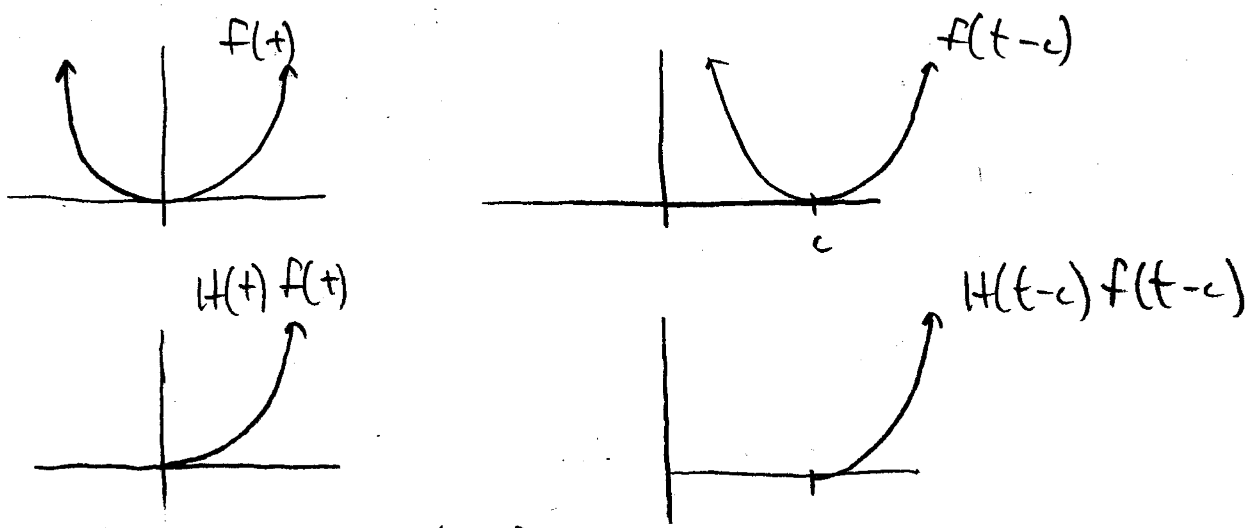
We'll soon see how to take the Laplace transform of this.

<u>Note</u>: $\mathcal{L}\{H_{ab}(t)\}(s) \int_a^b e^{-st}\,dt = \dfrac{e^{-as} - e^{-bs}}{s}$.

Recall that $\mathcal{L}\{e^{ct} f(t)\}(s) = F(s-c)$.

<u>Prop</u>: $\boxed{\mathcal{L}\{H(t-c) f(t-c)\}(s) = e^{-cs} F(s)}$.

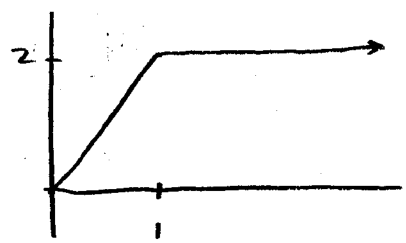
What does this mean? Suppose $f(t) = t^2$



Thus, $H(t-c) f(t-c)$ "shifts" $f(t)$ by c, and truncates it.

Since $f(t)$ is defined for only $t \geq 0$, we don't want $f(t-c)$ to be defined for $t < c$, so we use $H(t-c)\, f(t-c)$.

Ex Find the Laplace transform of $g(t) = \begin{cases} 2t & 0 \leq t < 1 \\ 2 & 1 \leq t < \infty. \end{cases}$



$$g(t) = 2t\, H_{01}(t) + 2\, H_1(t) =$$
$$= 2t[H(t) - H(t-1)] + 2[H(t-1)]$$
$$= 2t\, H(t) + (2-2t)\, H(t-1)$$
$$= 2t\, H(t) - 2(t-1)\, H(t-1)$$
$$\mathcal{L}\{g\} = 2\mathcal{L}\{t\, H(t)\} - 2\mathcal{L}\{(t-1)\, H(t-1)\}$$
$$= \frac{2}{s^2} - \frac{2}{s^2} e^{-s}$$

Practice:

- $\mathcal{L}\{(t-3)^2 H(t-3)\}(s) = \boxed{\frac{2}{s^3} e^{-3s}}$

  $f(t-3) = (t-3)^2 \Rightarrow f(t) = t^2, \quad F(s) = \frac{2}{s^3}$

- $\mathcal{L}\{t^2 H(t-3)\}(s) = e^{-3s} F(s) = \boxed{e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)}$

  $f(t-3) = t^2 \Rightarrow f(t) = f((t+3)-3) = (t+3)^2 = t^2 + 6t + 9$

  $F(s) = \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}$

- $\mathcal{L}\{e^{t-1} H(t-1)\}(s) = e^{-s} F(s) = \boxed{e^{-s} \frac{1}{s-1}}$

  $f(t-1) = e^{t-1} \Rightarrow f(t) = e^t, \quad F(s) = \frac{1}{s-1}$

- $\mathcal{L}\{e^{7-t} H(t-6)\} = e^{-6s} F(s) = \boxed{e^{1-6s} \frac{1}{1+s}}$

  $f(t-6) = e^{7-t} \Rightarrow f(t) = e^{7-(t+6)} = e^{1-t} = e \cdot e^{-t}, \quad F(s) = e\frac{1}{1+s}$

Ex: Find $G(s)$, where $g(t) = \begin{cases} 3 & 0 \le t < 4 \\ -5 & 4 \le t < 6 \\ e^{7-t} & 6 \le t < \infty \end{cases}$

Recall: $g(t) = 3\,H(t) - 8\,H(t-4) + 5\,H(t-6) + e^{7-t}H(t-6)$

$$G(s) = \frac{3}{s} - \frac{8}{s}e^{-4s} + \frac{5}{s}e^{-6s} + \frac{1}{s+1}e^{1-6s}$$

$$= \frac{3 - 8e^{-4s} + 5e^{-6s}}{s} + \frac{e^{1-6s}}{s+1}.$$

Ex: Solve the IVP $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 1$, where

$f(t) = \begin{cases} 2t & 0 \le t \le 1 \\ 2 & t > 1 \end{cases}$.

Recall: $f(t) = 2t\,H(t) - 2(t-1)\,H(t-1)$

and $F(s) = \frac{2}{s^2} - \frac{2e^{-s}}{s^2}$

Take $\mathcal{L}$ of both sides of the ODE:

$$[s^2Y - s\,y(0) - y'(0)] + Y = \frac{2 - 2e^{-s}}{s^2}$$

$$s^2Y - 1 + Y = \frac{2 - 2e^{-s}}{s^2}$$

$$(s^2+1)Y = \frac{2 - 2e^{-s}}{s^2} + 1$$

$$Y(s) = \frac{2 - 2e^{-s}}{s^2(s^2+1)} + \frac{1}{s^2+1}$$

Partial fractions $\Rightarrow$

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$= \frac{2}{s^2} - \frac{2}{s^2+1} - \frac{2e^{-s}}{s^2} + \frac{2e^{-s}}{s^2+1} + \frac{1}{s^2+1}$$

$$= \frac{2}{s^2} - \frac{2e^{-s}}{s^2} - \frac{1}{s^2+1} + \frac{2e^{-s}}{s^2+1}$$

$$y(t) = 2\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - 2\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) + 2\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+1}\right)$$

$$= 2t - 2(t-1)H(t-1) - \sin t + 2\sin(t-1)H(t-1)$$

$$= \left[2t - \sin t\right] + \left[2\sin(t-1) - 2(t-1)\right] H(t-1)$$

$$= \begin{cases} 2t - \sin t & 0 \le t < 1 \\ 2 + 2\sin(t-1) - \sin t & t \ge 1 \end{cases}$$

This is the particular sol'n to the ODE $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 1$.

Partial fraction tips:

Examples $\dfrac{x+3}{(x+2)^2(x-5)^2} \stackrel{?}{=} \dfrac{A}{(x+2)^2} + \dfrac{B}{(x-5)^2}$ Can we find A & B?

check: $\dfrac{A}{(x+2)^2}\dfrac{(x-5)^2}{(x-5)^2} + \dfrac{B}{(x-5)^2}\dfrac{(x+2)}{(x+2)} = \dfrac{(\quad)x^2 + (\quad)x + (\quad)}{(x-5)^2(x+2)}$

We have $\underline{\underline{2}}$ variables (A & B) but $\underline{\underline{3}}$ unknowns.

Can't solve. Need more variables. Try $Ax+C$.

$$\frac{(Ax+C)}{(x+2)}\frac{(x-5)^2}{(x-5)^2} + \frac{B}{(x-5)^2}\frac{(x+2)}{(x+2)} = \frac{(\quad)x^3 + (\quad)x^2 + (\quad)x + (\quad)}{(x-5)^2(x+2)}$$

Now we have $\underline{\underline{3}}$ variables and $\underline{\underline{4}}$ unknowns. Try $Bx+D$.

$$\frac{(Ax+C)}{(x+2)^2}\frac{(x-5)^2}{(x-5)^2} + \frac{(Bx+D)}{(x-5)^2}\frac{(x+2)^2}{(x+2)^2}$$

Numerator: $\overbrace{(A+B)}^{=0}x^3 + \overbrace{(-10A + 4B + C + D)}^{=0}x^2$

$+ \underbrace{(25A + 4B - 10C + 4D)}_{=1}x + \underbrace{(25C + 4D)}_{=3}$

Now, use a calculator/computer to solve the system:

$$\begin{cases} A+B=0 \\ -10A+4B+C+D=0 \\ 25A+4B-10C+4D=1 \\ 25C+4D=3 \end{cases}$$

Example: $\dfrac{2}{(s^2+9)(s-1)^2} = \dfrac{(As+C)}{s^2+9}\dfrac{(s^2-2s+1)}{(s-1)^2} + \dfrac{(Bs+D)}{(s-1)^2}\dfrac{(s^2+9)}{s^2+9}$

$$= \underbrace{(A+B)}_{=0}s^3 + \underbrace{(-2A+B+C+D)}_{=0}s^2 + \underbrace{(A+9B-2C)}_{=0}s + \underbrace{(C+9D)}_{=2}$$

$$\begin{cases} A+B=0 \\ -2A+B+C+D=0 \\ A+9B-2C=0 \\ C+9D=2 \end{cases}$$

Exercise: Compute $\mathcal{L}^{-1}\left(\frac{s}{(s-1)^2}\right)$.

Method 1: let $g(s) = \dfrac{s}{(s-1)^2} = \dfrac{s-1}{(s-1)^2} + \dfrac{1}{(s-1)^2} = \dfrac{1}{(s-1)} + \dfrac{1}{(s-1)^2}$ (*)

If $F(s) = \frac{1}{s} + \frac{1}{s^2}$, then $f(t) = 1+t$

But $G(s) = F(s-1)$. Thus $g(t) = e^t f(t) = \boxed{e^t(1+t)}$

Method 2: $\dfrac{s}{(s-1)^2} = \dfrac{A}{(s-1)}\dfrac{(s-1)}{(s-1)} + \dfrac{B}{(s-1)^2} = \dfrac{(\underbrace{A}_{=1})s + (\underbrace{B-A}_{=0})}{(s-1)^2}$

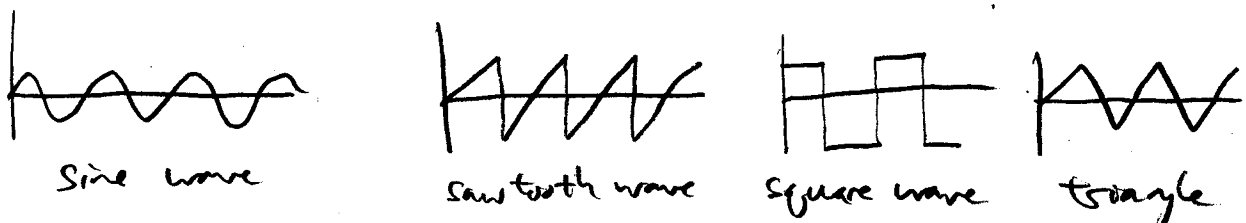
Thus $A=1$ & $B=1$ $\Rightarrow$ $g(s) = \dfrac{1}{s-1} + \dfrac{1}{(s-1)^2}$.

Now we're "back in Method 1," at (*).

Periodic Forcing terms

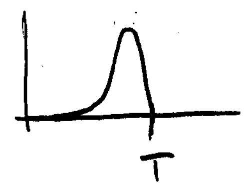
- Suppose $f(t)$ is periodic. We want to compute $\mathcal{L}(f(t))$.

e.g.,



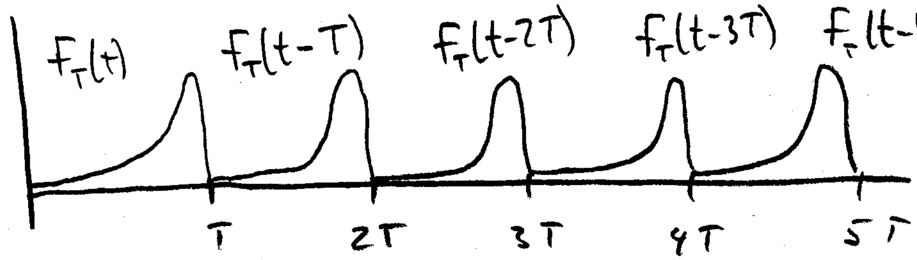
- Suppose $f_T(t)$ is defined on $0 \le t < T$.

"The window"



Then the function $f(t) = \begin{cases} f_T(t) & 0 \le t < T \\ f_T(t-kT) & kT \le t < (k+1)T \end{cases}$

is periodic:

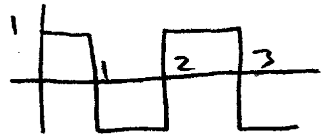


<u>Claim</u>: $\mathcal{L}(f(t))(s) = \dfrac{F_T(s)}{1-e^{-Ts}} = \dfrac{\int_0^T f(t)\, e^{-st}\, dt}{1-e^{-Ts}}$

<u>Proof</u>: (Key idea: If $|x|<1$, then $1+x+x^2+\ldots = \boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$)

$$f(t) = \sum_{k=1}^{\infty} f_T(t-kT) = \sum_{k=1}^{\infty} f_T(t-kT)\, H(t-kT)$$

$$\begin{aligned} F(s) &= \sum_{k=1}^{\infty} \mathcal{L}\{ f_T(t-kT)\, H(t-kT)\}(s) \\ &= \sum_{k=1}^{\infty} e^{-kTs} F_T(s) = F_T(s) \sum_{k=1}^{\infty} e^{-kTs} \\ &= F_T(s) \sum_{k=1}^{\infty} \left(e^{-sT}\right)^k = \boxed{F_T(s) \frac{1}{1-e^{-sT}}} \end{aligned}$$

Example: Let $f(t) =$ 

Period $T = 2$

Solve the IVP $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 0$.

First, compute $F(s)$.

$$f_T(t) = H_{01}(t) - H_{12}(t) = [H(t) - H(t-1)] - [H(t-1) - H(t-2)]$$

$$= H(t) - 2H(t-1) + H(t-2)$$

$$F_T(s) = \mathcal{L}(H(t)) - 2\mathcal{L}(H(t-1)) + \mathcal{L}(H(t-2))$$

$$= \frac{1}{s} - \frac{2}{s}e^{-s} + \frac{1}{s}e^{-2s} = \boxed{\frac{(1-e^{-s})^2}{s}}$$

Now, $F(s) = F_T(s)\dfrac{1}{1-e^{-2s}} = \dfrac{F_T(s)}{(1-e^{-s})(1+e^{-s})}$

$$= \frac{(1-e^{-s})(1-e^{-s})}{s(1-e^{-s})(1+e^{-s})} = \boxed{\frac{(1-e^{-s})}{s(1+e^{-s})}}$$

Back to IVP: $\mathcal{L}(y'') + \mathcal{L}(y) = F(s)$

$$[s^2 Y - s\,y(0) - y'(0)] + Y = \frac{1-e^{-s}}{s(1+e^{-s})}$$

$$(s^2+1)Y = \frac{1-e^{-s}}{s(1+e^{-s})} \Rightarrow \boxed{Y(s) = \frac{1-e^{-s}}{s(s^2+1)(1+e^{-s})}}$$

Simplify this: $Y(s) = \underbrace{\dfrac{1}{s(s^2+1)}}_{= \frac{1}{s} - \frac{s}{s^2+1}} \cdot \underbrace{\dfrac{1-e^{-s}}{1+e^{-s}}}_{\text{need to simplify this.}}$

$$\frac{1-e^{-s}}{1+e^{-s}} = \frac{-(1+e^{-s})}{1+e^{-s}} + \underbrace{\frac{2}{1+e^{-s}}}$$

$$= 2\left(\frac{1}{1-(-e^{-s})}\right) = 2\sum_{n=0}^{\infty}(-1)^n e^{-ns}$$

Note: $\frac{1}{1-x} = \sum_{n=1}^{\infty} x^n$

and $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$

Thus, $Y(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right)\left(-1 + 2\sum_{n=0}^{\infty}(-1)^n e^{-ns}\right)$

$$= \underbrace{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)}\left(1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-ns}\right)$$

Call this $F(s)$. Note that $f(t) = 1 - \cos t$.

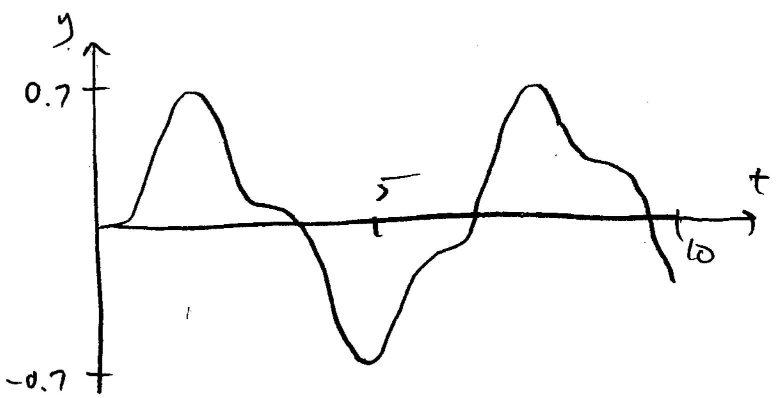
use $\mathcal{L}\{f(t-n)\,H(t-n)\}(s) = e^{-ns}F(s)$

So, $Y(s) = F(s) + 2\sum_{n=1}^{\infty}(-1)^n e^{-ns}F(s)$

$$y(t) = f(t) + 2\sum_{n=1}^{\infty}(-1)^n f(t-n)\,H(t-n)$$

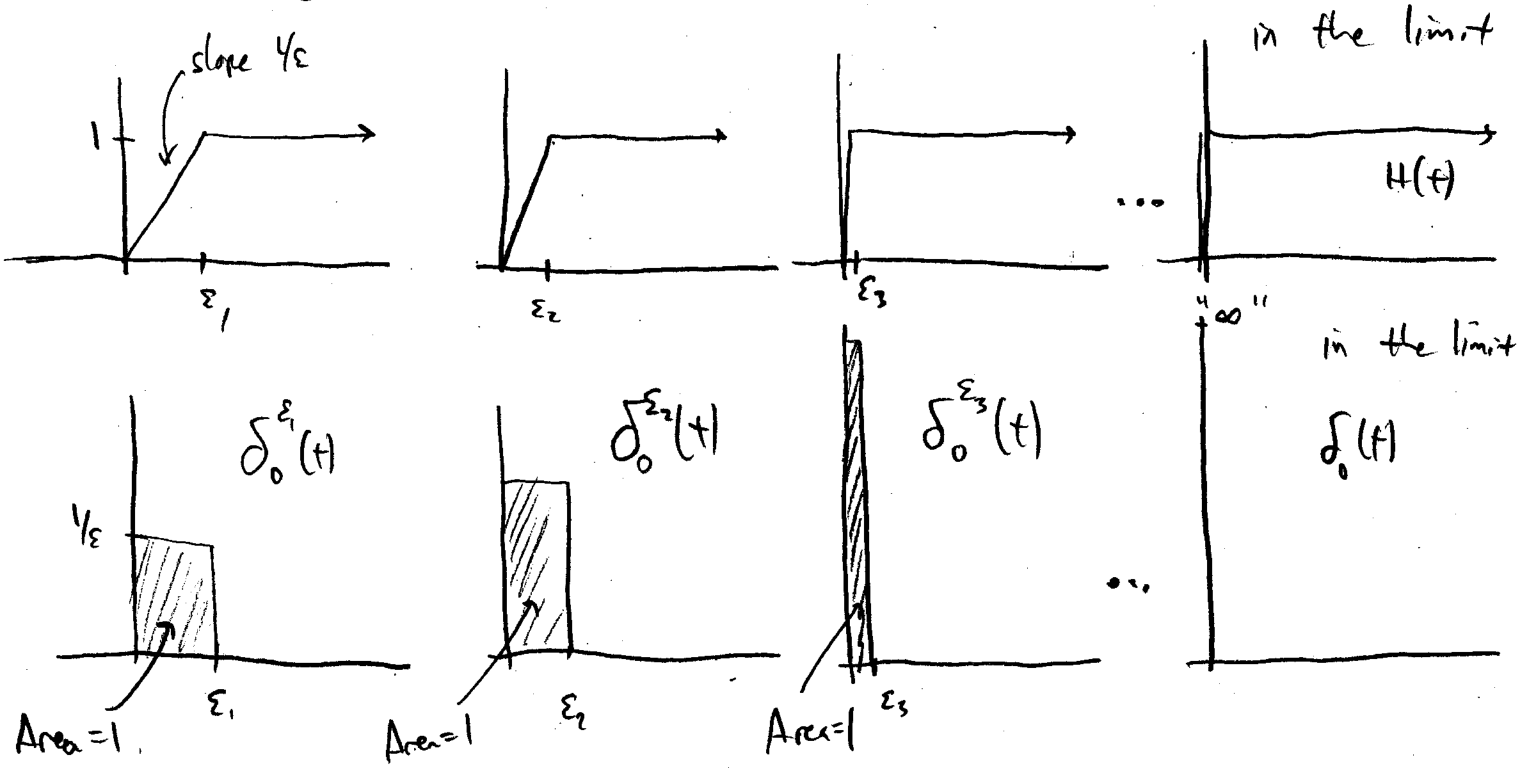
$$\boxed{y(t) = (1-\cos t)\,H(t) + 2\sum_{n=1}^{\infty}(-1)^n\left[1-\cos(t-n)\right]H(t-n)}$$

This is a superposition of infinite waves.



<u>Question</u>: What is the "derivative" of the Heavyside function?

Technically, it's not defined, but what "should" it be?



<u>Def</u>: The <u>delta function</u> is $\delta_p(t) = \lim_{\varepsilon \to 0} \delta_p(t)$.

Technically, it's not a function. (Engineers like to cheat!)

<u>But it's useful</u>:

- $\int_0^\infty \delta_p(t)\, f(t)\, dt = f(p)$
- $\mathcal{L}(\delta_p)(s) = e^{-sp}$
- $\mathcal{L}(\delta_0)(s) = 1$ (so now you can take the inverse Laplace trans. of a constant, or exponential).

- This models a unit impulse force (Finite force over an infintesimal time interval).

e.g., Exerting a force by hitting something with a hammer.

<u>Example:</u> Solve the IVP $y'' + 2y' + 2y = \delta_0(t)$ $\quad y(0) = 0, \ y'(0) = 0.$

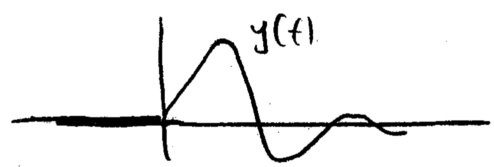
First, take Laplace transform: $\mathcal{L}(y'' + 2y' + 2y) = \mathcal{L}(\delta_0(t))$

$$(s^2+2s+2)\,Y = 1 \quad \Rightarrow \quad Y(s) = \frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}$$

$\Rightarrow \ y(t) = e^{-t}\sin t.$

<u>Note:</u> $y(0) = 0$, but $y'(0) = 1$. How to "Fix" this?

Define $y(t) = \begin{cases} 0 & t \le 0 \\ e^{-t}\sin t & t \ge 0 \end{cases}$



This isn't even differentiable at $t = 0$, so technically, $y'(0)$ isn't defined. But $\lim_{t \to 0^-} = 0$, and that's "good enough."

<u>Skip to Chapter 9:</u>

Recall ch 2: $\boxed{x' = ax \quad x(0) = C}$ has sol'n $\boxed{x(t) = Ce^{at}}$

Ch 9: Systems of ODEs:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 \\ x_2' &= a_{21}x_1 + a_{22}x_2 \end{aligned} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \vec{x}'$$

Further, suppose that $\begin{aligned} x_1(0) &= v_1 \\ x_2(0) &= v_2 \end{aligned} \Rightarrow \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \bar{v}$

we can write this as $\boxed{\vec{x}\,' = A\vec{x}, \quad \vec{x}(0) = \vec{v}}$

So clearly, this must have sol'n $\boxed{\vec{x}(t) = e^{At}\vec{v}}$!

What?! Does this even make sense?

How do we even define e^{At}?

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

so $$e^{At} = 1 + t + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + \frac{A^5t^5}{5} + \dots = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$$

<u>where are we going from here?</u> (7 weeks remaining).

- Solving ODEs with power series
- Some basic linear algebra (my way)
- Fourier series (coolest application of linear algebra ever!)

  <u>Big idea</u>: For any periodic function $f(t)$, we can write
  $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \quad \text{uniquely!}$$
- Partial differential equations (PDEs).

  e.g. $u(x,t)$: $\quad u_t = u_{xx}, \quad u_x(x,0) = 0, \quad u(x,10) = 0, \quad u_t(0,x) = e^x$

  Including:
  - what they model
  - How to solve them
  - Laplace transforms: turn PDEs into ODEs.
  - Fourier transforms: similar to Laplace transforms.