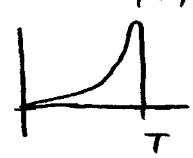
MTHSC 208, WEEK 9

Week 8 summary

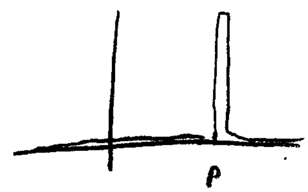
- $\mathcal{L}\{H(t-c)\, f(t-c)\}(s) = e^{-cs} F(s)$



- If $f(t)$ is periodic with period T and "window" $F_T(t)$

  then $\mathcal{L}\{f(t)\}(s) = \dfrac{F_T(s)}{1-e^{-sT}} = F_T(s)\left[1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \dots\right]$

- Delta function $\delta_p(t) = \begin{cases} \text{"}\infty\text{"} & t = p \\ 0 & t \neq p \end{cases} = \lim_{\varepsilon \to 0} \delta_p^{\varepsilon}(t)$



  (not really a function)

  * Allows us to take $\mathcal{L}^{-1}$ of a constant

  $\mathcal{L}(\delta_p(s)) = e^{-sp}$, $\mathcal{L}(\delta_0(s)) = 1$

  * $\int_0^\infty \delta_p(t)\, f(t)\, dt = f(p)$, $\int_0^\infty \delta_p(t)\, dt = 1$

  * Models a unit impulse force (Finite force applied over an instantaneous time period).

Back to solving ODES.

Motivation: Sometimes we were able to guess what the general solution of an ODE looks like.

e.g.:

- $y' = ky$ $\qquad y(t) = Ce^{kt}$
- $y' = k(A-y)$ $\qquad y(t) = A + Ce^{-kt}$
- $y'' = -k^2 y$ $\qquad y(t) = A\cos kt + B\sin kt$

- $ay'' + by' + cy = 0$ $\qquad y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
- $ay'' + by' + cy = e^{rt}$ $\qquad y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + A e^{rt}$
- $ay'' + by' + cy = \sin 3t$ $\qquad y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + (A \cos 3t + B \sin 3t)$

$\vdots$

What about this one: • $x^2 y'' + xy' - y = 0$

What's a good guess? Try $y(x) = x^r$ (why?)

$y'(x) = r x^{r-1} \qquad y''(x) = r(r-1)x^{r-2}$

Plug back in:
$$\begin{aligned} x^2 y'' + xy' - y &= x^2 r(r-1)x^{r-2} + x r x^{r-1} - x^r \\ &= r(r-1)x^r + r x^r - x^r \\ &= x^r(r(r-1) + r - 1) \\ &= x^r(r^2 - 1) = 0 \implies r = \pm 1 \end{aligned}$$

We found two solutions! $y_1(x) = x$, $y_2(x) = x^{-1}$

Thus, the general solution is $\boxed{y(x) = C_1 x + C_2 x^{-1}}$

Question: Does this always work? What could go wrong?

What if instead, we had: • $x^2 y'' + xy' + y = 0$?

Again, try $y(x) = x^r$.

$\vdots$

End up with $x^r(r^2 + 1) = 0$. uh oh. $y(x) = x^i$ or x^{-i}

What does x^i even mean?

$$x^i = (e^{\ln x})^i = e^{i \ln x} = 1 + i\ln x + \frac{i^2(\ln x)^2}{2!} + \frac{i^3(\ln x)^3}{3!} + \dots$$

$$= 1 + i\ln x - \frac{(\ln x)^2}{2!} + \frac{i(\ln x)^3}{3!} + \frac{(\ln x)^4}{4!} + \dots$$

Let's use Euler's equation: $e^{ix} = \cos x + i \sin x$

$$y_1 = e^{i \ln|x|} = \cos(\ln|x|) + i \sin(\ln|x|)$$

$$y_2 = e^{-i \ln|x|} = \cos(\ln|x|) - i \sin(\ln|x|)$$

Note: $\frac{1}{2}\left[y_1(x) + y_2(x)\right] = \cos(\ln|x|)$ is a sol'n

$\frac{1}{2i}\left[y_1(x) - y_2(x)\right] = \sin(\ln|x|)$ is a sol'n

So the general sol'n is $\boxed{y(t) = C_1 \cos(\ln|x|) + C_2 \sin(\ln|x|)}$.

* In general, if the roots are $r_{1,2} = a \pm bi$, the general sol'n will be

$$\boxed{\begin{aligned} y(x) = {} & C_1 e^{ax}\cos(b\ln|x|) \\ & + C_2 e^{ax}\sin(b\ln|x|). \end{aligned}}$$

What else could go wrong?

- $x^2 y'' - xy' + y = 0.$ Again, assume $y(x) = x^r$,

$y'(x) = r x^{r-1}$, $y''(x) = r(r-1)x^{r-2}$

Plug back in:

$$x^2\left[r(r-1)x^{r-2}\right] - x\left[r x^{r-1}\right] - x^r = 0$$

$$x^r\left[r(r-1) - r + 1\right] = x^r(r-1)^2 = 0 \qquad \text{Uh oh....}$$

$y(x) = x$ is a solution. But we need another.

Try $y(x) = v(x) \cdot x$, and solve for $v(x)$.

$y' = v'x + v$, $\quad y'' = v''x + 2v'$

Plug back in: $x^2\left[v''x + 2v'\right] - x\left[v'x + v\right] + vx = 0$

$x^3 v'' + x^2 v' = 0$ divide by x^2

$x v'' + v' = 0$ let $w = v'$

$x w' + w = 0$

$\frac{dw}{dx} = -\frac{w}{x} \qquad \int \frac{dw}{w} = -\int \frac{dx}{x}$

$\ln|w| = -\ln|x| + C$

$\ln|w| + \ln|x| = C$

$\ln|xw| = C \Rightarrow \ln|xv'| = C.$

Or just $xv' = C_1$ (for some other const. C_1).

$x \frac{dv}{dx} = C \Rightarrow \int dv = C\int \frac{dx}{x} \Rightarrow v(x) = C_1 \ln|x| + C_2$

Since $y(x) = v(x)\, x$, $\quad y(x) = C_1 x \ln|x| + C_2 x$.

Sol'n 1: $y_1(x) = Cx$, $\quad$ sol'n 2: $y_2(x) = C_1 x \ln|x| + C_2 x$

General sol'n: $\boxed{y(x) = C_1 x + C_2 x \ln|x|}$

Here, we say that $\{x \ln|x|, x\}$ is a basis for the solution space of this ODE.

Note: $\{x, x\ln|x| + x\}$ is also a basis. But the other one is "better."

Let's make things harder

- $y'' - 4xy' + 12y = 0$

What do we assume the solution will be?

Note: $y(x) = x^r$ won't work!

Why not? If $y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$,

then $y'' - 4xy' + 12y = r(r-1)\underline{x^{r-2}} + 4r\underline{x^r} + 12\underline{x^r} = 0$

Maybe try $y(x) = ax^r + bx^{r-2}$?

Then, we'll get $(\qquad)x^{r-4} + (\qquad)x^{r-2} + (\qquad)x^r = 0.$

This will give us a sol'n, since there are 3 equations and three unknowns (a, b, r).

But, it'll only get us one sol'n.

Better method: Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Why? Because most "nice" functions have a Taylor Series expansion, so let's find that.

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, \qquad y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Plug back in: $$\underbrace{\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}}_{\text{re-write this so we can combine terms.}} - 4\sum_{n=0}^{\infty} n a_n x^n + 12\sum_{n=0}^{\infty} a_n x^n = 0 \quad (*)$$

Let $m = n-2$: $$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{m=-2}^{\infty} (m+2)(m+1) a_{m+2} x^m$$

$$(\text{why?}) \quad = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m$$

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Now, switch back to using n:

$(*)$ becomes $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 4\sum_{n=0}^{\infty} n\,a_n x^n + 12\sum_{n=0}^{\infty} a_n x^n = 0$

$$\sum_{n=0}^{\infty}\Big[\underbrace{(n+2)(n+1)a_{n+2}+(12-4n)a_n}_{\text{set }=0.}\Big]x^n = 0$$

$\Rightarrow \quad (n+2)(n+1)a_{n+2}+(12-4n)a_n = 0 \quad$ for all n.

$\Rightarrow \quad a_{n+2} = \dfrac{4n-12}{(n+2)(n+1)}a_n. \quad$ We now have a <u>recurrence relation</u>.

<u>Note</u>: Choose <u>any</u> a_0. <u>All</u> the even a_n's are determined.

Choose <u>any</u> a_1. <u>All</u> the odd a_n's are determined.

Thus, the even and odd terms are independent of each other, i.e., $\left\{\sum_{n=0}^{\infty}a_{2n}x^{2n},\ \sum_{n=0}^{\infty}a_{2n+1}x^{2n+1}\right\}$ is a <u>basis</u> for the solution space.

Let's compute the first few terms (in terms of a_0 & a_1).

$a_2 = -\frac{12}{2}a_0 = -6a_0 \qquad\qquad a_3 = \frac{-8}{3!}a_1 = \frac{-4}{3}a_1$

$a_4 = \dfrac{-4}{4\cdot 3}a_2 = \dfrac{(-4)(-12)}{4!}a_0 = \frac{2}{2}a_0 \qquad\qquad a_5 = 0$

$a_6 = \dfrac{(4)(-4)(-12)}{6!}a_0 = \frac{4}{15}a_0 \qquad\qquad a_7 = 0$

$a_9 = 0$

$\vdots$

$a_8 = \dfrac{(12)(4)(-4)(-12)}{8!} = \frac{2}{35}a_0$

and so on... $\quad a_n = \dfrac{(4\cdot 0-12)(4\cdot 2-12)(4\cdot 4-12)\cdots[4(n-2)-12]}{n!} \quad (**)$

The general sol'n is $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Note: IF $a_1 \neq 0$, then the odd terms are

$$\sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} = a_1 x + a_3 x^3 = a_1 x - \tfrac{4}{3} a_1 x^3 = a_1\left(x - \tfrac{4}{3}x^3\right).$$

The even terms are $\sum_{n=0}^{\infty} a_{2n} x^{2n}$,

Thus, a basis for the sol'n space is

$$\left\{ \sum_{n=0}^{\infty} a_{2n} x^{2n}, \ \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} \right\} = \left\{ \sum_{n=0}^{\infty} a_{2n} x^{2n}, \quad x - \tfrac{4}{3} x^3 \right\}$$

Where the a_n's are given by $(**)$.

Summary: To solve $y'' - 4xy' + 12y = 0$, we

- Assumed the sol'n was of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$
- Plugged $y(x)$ back into the ODE
- Combined into a single sum $\sum_{n=0}^{\infty} [\qquad\qquad] x^n = 0$
- Set the coefficients equal to zero to get a recurrence relation for a_{n+2}.

Note: Pick $a_0 \neq 0$ & $a_1 \neq 0$ at will. Then $y_0(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n}$ (even terms) and $y_1(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n+1}$ (odd terms) formed a basis for the solution space.

i.e., every solution $y(x)$ can be written uniquely as a linear combination $y(x) = C_0 y_0(x) + C_1 y_1(x)$.