MTHSC 208, WEEK 10

Week 9 summary:

- $x^2y'' + axy' + by = 0$ has sol'n $y(x) = x^r$. Solve for r.

Case 1: $r_1 \neq r_2$ real: $y(x) = C_1x^{r_1} + C_2x^{r_2}$

Case 2: $r_1 = r_2$ real $y_1(x) = x^r$, $y_2(x) = v(x)\,x^r$. Solve for $v(x)$.

Case 3: $r_{1,2} = a \pm bi$. $y_1(x) = x^{a+bi}$, $y_2(x) = x^{a-bi}$

Use Euler's formula to get

$y(x) = x^a\left(C_1\cos\left(b\ln|x|\right) + C_2\sin\left(b\ln|x|\right)\right)$.

- $y'' + axy' + by = 0$ has sol'n $y(x) = \sum_{n=0}^{\infty} a_n x^n$

Plug back in, get $\sum_{n=0}^{\infty}\left[\qquad\qquad\right]x^n = 0$.

recurrence relation. $a_{n+2} = \dfrac{\quad}{(n+2)(n+1)}\,a_n$.

$y(0) = a_0$: Picking a_0 determines the even terms.

$y'(0) = a_1$: Picking a_1 determines the odd terms.

Review of power series:

A power series centered at x_0 is a series of the form $\sum_{n=0}^{\infty} a_n(x-x_0)^n = \lim_{N\to\infty} \underbrace{\sum_{n=0}^{N} a_n(x-x_0)^n}_{\text{"partial sum"}}$

A power series converges at x if the sequence of partial sums converges. Otherwise it diverges.

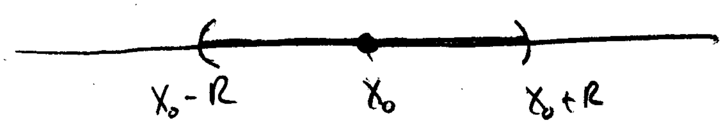
Ex: $\lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!} x^n = e^x$ for all x.

Non-ex: $\lim_{N\to\infty} \sum_{n=0}^{N} (-1)^n x^n$ does not converge for $x=1$, because the sequence of partial sums of $\sum_{n=0} (-1)^n x^n = 1-1+1-1+-\dots$ is $1, 0, 1, 0, 1, 0, \dots$

key point: Sometimes a series won't converge everywhere.

Ex: $y(x) = \sum_{n=0}^{\infty} x^n$:
- Converges to $\frac{1}{1-x}$ if $|x|<1$
- Diverges if $|x| \geq 1$.

We say that the radius of convergence is the largest number R such that if $|x-x_0| < R$, $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges at x. If it converges for all x, we say $R=\infty$



Ex: $y(x) = \sum_{n=0}^{\infty} x^n$ $(a_n = 1, \; x_0 = 0)$ has $R = 1$.

$y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ has $R = \infty$. (since it converges to e^x for all x).

What do we mean by "centered at x_0"?

Compare to $f(x) = x^2$ and $f(x-x_0) = (x-x_0)^2$

This just "shifts" the graph by x_0.
Alternatively, it's a change of variables: let $u = x - x_0$.

<u>Taylor's theorem</u>: $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ where $a_n = \frac{f^{(n)}(x_0)}{n!}$

<u>Why we care</u>: For large N, $\sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ is a <u>very good</u> approximation of $f(x)$... as long as $x \approx x_0$.

<u>Ex</u>: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

<u>Q</u>: Is $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!}$?

<u>A</u>: Yes... as long as $x \approx 0$.

<u>Note</u>: It would be a <u>horrible</u> approximation if $x \approx 1{,}000{,}000$

<u>Why</u>? The first discarded term, $\frac{(1{,}000{,}000)^9}{9!}$ is huge!

Suppose we wanted to find e^x, if $x \approx 1{,}000{,}000$. We'd use:

$$e^x \approx e^{1{,}000{,}000} + \frac{e^{1{,}000{,}000}}{1!}(x-1{,}000{,}000) + \frac{e^{1{,}000{,}000}}{2!}(x-1{,}000{,}000)^2 + \dots + \frac{e^{1{,}000{,}000}}{8!}(x-1{,}000{,}000)^8$$

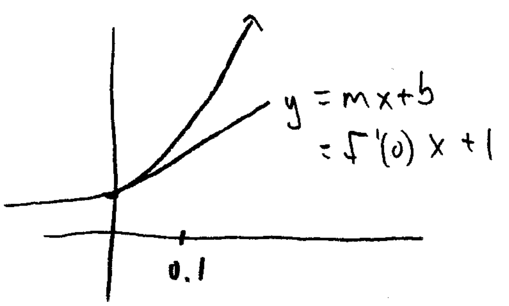
These infinite sums are the same, but <u>not</u> the partial sums.

What's going on geometrically?

[http://mathdemos.gcsu.edu/mathdemos/TaylorPolynomials/]

<u>Ex 1</u>: Recall single variable calculus:

"Use the tangent line to approximate $e^{0.1}$"



The first 2 terms of the Taylor series at $x=0$.

This is a linear approx (best fit deg-1 poly).

We could also do a quadratic, or cubic approx (best fit deg-2 & 3 polys).

[SHOW DEMO]

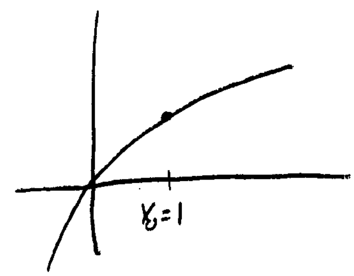
Note: These are only good approximations for $x \approx 0$ (because we're using a power series centered at $x_0 = 0$).

Ex 2: Visualization of the radius of convergence.

Let $f(x) = \ln(x+1)$.

Taylor series: $f(x) = \sum_{n=1}^{\infty} (-1)^n x^n$. Radius of conv. $R = 1$.

Thus, no matter how far we go out, this will not be a good approx. for $\ln(x+1)$ as long as $|x| \geq 1$.



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Computing R:

Method 1: Directly. (This is rare).

Ex: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. When does this converge?

Look at the partial sums.

$$S_N(x) = \sum_{n=0}^{N} x^n = 1 + x + x^2 + \dots + x^N.$$

By def'n, our series converges if $\lim_{N\to\infty} S_N(x) = \frac{1}{1-x}$.

When does this happen?

If $S_N(x) \longrightarrow \frac{1}{1-x}$, then $(1-x)\, S_N(x) \longrightarrow 1$.

But $(1-x)S_N(x) = (1-x)(1+x+x^2+\dots+x^N)$

$= (1+x+\dots+x^N) - (x+x^2+x^3+\dots+x^{N+1})$

$= 1-x^{N+1}$ (telescopes!)

And clearly, $1-x^{N+1} \longrightarrow 1$ iff $|x|<1$. So $\boxed{R=1}$.

Method 2: Ratio test:

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|},$$ if this limit exists.

Ex: Taylor series for $\ln(x+1) = \sum_{n=1}^{\infty} (-1)^{n+1} x^n = x - x^2 + x^3 - x^4 +/- \dots$

So, $|a_n| = 1$ for all $n \geq 1$. $\Rightarrow R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \frac{1}{1} \Rightarrow \boxed{R=1}$

Ex: $y(x) = \sum_{n=0}^{\infty} \frac{1}{3^n} x^n$. $a_n = \frac{1}{3^n}$, $\lim_{n\to\infty} \frac{|1/3^n|}{|1/3^{n+1}|} = 3 \Rightarrow \boxed{R=3}$

Ex: $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ $a_n = \frac{1}{n!}$, $\lim_{n\to\infty} \frac{|1/n!|}{|1/(n+1)!|} = n \to \infty \Rightarrow \boxed{R=\infty}$

Method 3: Comparison test:

"power series with smaller coefficients have larger radii of conv."

Say $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ has radius of conv. R_1

and $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ has radius of conv. R_2.

If $|a_n| \leq |b_n|$ for all n, then $R_1 \geq R_2$

If $|a_n| \geq |b_n|$ for all n, then $R_1 \leq R_2$.

Ex: $1 + 0x - \frac{1}{2!}x^2 + 0x^3 + \frac{1}{4!}x^4 +/- \dots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$

$$\text{Compare to: } \overset{a_0}{1} + \overset{a_1}{1}x + \overset{a_2}{\frac{1}{2!}}x^2 + \overset{a_3}{\frac{1}{3!}}x^3 + \overset{a_4}{\frac{1}{4!}}x^4 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

$$y(x) = \underset{b_0}{1} + \underset{b_1}{0}x_1 - \underset{b_2}{\frac{1}{2!}}x^2 + \underset{b_3}{0}x^3 + \underset{b_4}{\frac{1}{4!}}x^4 +/- \dots = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$$

We know $R_a = \infty$ since it's just e^x, or by ratio test

Since $|a_n| \geq |b_n|$ for all n, $R_b \geq R_a = \infty \Rightarrow \boxed{R_b = \infty}$

Note: Sometimes we can't find R, but we can bound R.

Back to ODEs:

The power series method really does come up in practice!!

- Hermite's diff eq: $y'' - 2xy' + 2py = 0$

  Used for treating simple harmonic oscillators in quantum mech.

- Legendre's diff eq: $(1-x^2)y'' - 2xy' + p(p+1)y = 0$

  Used for treating spherically symmetric potentials in theory of Newtonian gravitation, and in electricity & magnetism (E & M)

- Bessel's equation: $x^2y'' + xy' + (x^2 - p^2)y = 0$

  Used for analyzing vibrations of a circular drum.

- Chebyshev's equation: $(1-x^2)y'' - xy' + p^2y = 0$.