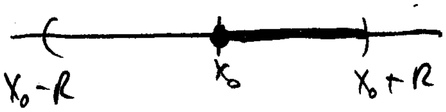
MTHSC 208, WEEK 11

Week 10 summary:

- A power series centered at x_0 is a function $\sum_{n=0}^{\infty} a_n (x-x_0)^n$.

  The radius of convergence R of a power series is the maximal R such that $|x-x_0| < R \Rightarrow \sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges.



- Ratio test: $R = \lim_{n\to\infty} \dfrac{|a_n|}{|a_{n+1}|}$, if this limit exists.

  Comparison test: Given $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ and $\sum_{n=0}^{\infty} b_n (x-x_0)^n$, then if $|a_n| \le |b_n|$, $R_a \geq R_b$.

Regular vs. singular points:

Def: A function $f(x)$ is real analytic at x_0 if
$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ for some $R > 0$.

i.e, real analytic @ x_0 $\Longleftrightarrow$ has a power series @ x_0.

Def. Consider the ODE $y'' + P(x)y' + Q(x)y = 0$.

* A point x_0 is an ordinary point if $P(x)$ & $Q(x)$ are real analytic at x_0
* If x_0 is not ordinary, then it is a singular point.

* If x_0 is singular, then it is <u>regular</u> if $(x-x_0)P(x)$ <u>and</u> $(x-x_0)^2 Q(x)$ are real analytic.

<u>Note</u>: Usually, if $f(x)$ isn't real analytic at x_0, then it's not defined at x_0. $\left(\text{e.g., } f(x) = \frac{1}{x},\ x_0 = 0\right)$.

<u>Why we care</u>:

(real analytic at x_0)

<u>Theorem of Frobenius</u>: Consider an ODE $y'' + P(x)y' + Q(x)y = f(x)$

- If x_0 is an <u>ordinary point</u>, and P, Q, F have radii of convergence R_P, R_Q, R_F, respectively, then there is a <u>power series solution</u> $y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, with $R = \min\{P, Q, F\}$.
- If x_0 is a <u>regular singular point</u>, and $(x-x_0)P(x)$, $(x-x_0)^2 Q(x)$, and $f(x)$ have radii of conv. R_P, R_Q, R_F resp, then there is a <u>generalized power series solution</u> $y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n$, for some constant r (possibly a fraction, or even complex).

<u>Note</u>: If x_0 is an irregular singular pt, then we're out of luck.

<u>Example</u>: Consider $y'' + x^2 y - 4y = 0$. $P(x) = x^2$, $Q(x) = -4$.

$P(x)$ & $Q(x)$ are real analytic for all x_0, with radii of conv. ∞.

Thus by Frobenius, there is a solution $\sum_{n=0}^{\infty} a_n (x - x_0)^n$, valid for all x (i.e, $R = \infty$).

<u>Example</u>: $y'' - \frac{x}{1-x^2} y' + \frac{p^2}{1-x^2} y = 0$. $P(x) = \frac{-x}{1-x}$, $Q(x) = \frac{1}{1-x^2}$.

$P(x)$ & $Q(x)$ are real analytic at $x = 0$:

$$Q(x) = \frac{1}{1-(x^2)} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \cdots$$

$$P(x) = \frac{-x}{1-x^2} = -x \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} -x^{2n+1} = -x - x^3 - x^5 - x^7 - \cdots$$

<u>Note</u>: $R_P = R_Q = 1$.

Thus, by Frobenius, there is a solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ with $R = 1$. (You'll find it on HW #16).

<u>Example</u>: $x^5 y'' + y' + y = 0$.

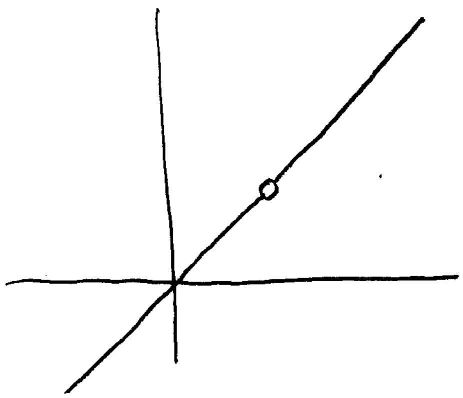
Write as $y'' + \frac{1}{x^5} y' + \frac{1}{x^5} y = 0$, $P(x) = \frac{1}{x^5}$, $Q(x) = \frac{1}{x^5}$.

$x_0 = 0$ is an irregular singular point, since $x P(x) = \frac{1}{x^4}$ isn't defined at $x_0 = 0$.

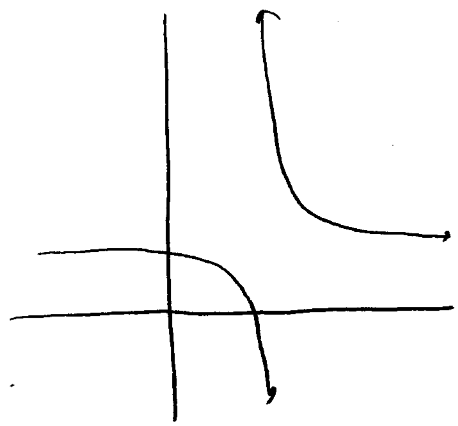
Frobenius does not guarantee a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

But we could find one of the form $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ if we wanted to. (B/c $x_0 = 1$ is regular).

Analogy: $f(x) = \frac{x(x-2)}{(x-2)}$ $\qquad$ $g(x) = \frac{x(x-2)}{(x-2)^2}$



This singularity is "fixable"



This singularity is "unfixable."

Example: $2xy'' + y' + y = 0$.

Write as $y'' + P(x)y' + Q(x)y = 0$, $\quad P(x) = \frac{1}{2x}$, $\quad Q(x) = \frac{1}{2x}$

$x_0 = 0$ is a regular singular point, since $x\,P(x) = \frac{1}{2}$, and

$x^2 Q(x) = \frac{1}{2}x$ are real analytic.

By Frobenius, there is a solution $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$.

We'll find it the same way as before:

$$y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1}$$

Plug back into the ODE:

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$= x^r \left[\sum_{n=0}^{\infty} (2n+2r-1)(n+r) a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \right] = 0.$$

Shift indices up by one (let $m=n-1$, or just do in your head)

$$\cancel{x^r}\left[\sum_{n=-1}^{\infty}(2n+2r+1)(n+r+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_nx^n\right]=0$$

← one extra term!

$$=\underbrace{(2r-1)ra_0x^{-1}}_{\text{set }=0}+\sum_{n=0}^{\infty}\underbrace{\left[(2n+2r+1)(n+r+1)a_{n+1}+a_n\right]}_{\text{set }=0}x^n=0.$$

$$\Downarrow$$

$$(2r-1)r=0 \quad \text{"indicial equation"}$$

$$\Downarrow$$

$$\boxed{r=0 \text{ or } r=\tfrac{1}{2}}$$

$$\Downarrow$$

$$\boxed{a_{n+1}=\frac{-1}{(2n+2r+1)(n+r+1)}a_n}$$

"recurrence relation"

We now have two solutions:

$\underline{r=0}$: $\quad y_0(x)=\sum_{n=0}^{\infty}a_nx^n \qquad a_{n+1}=\dfrac{-1}{(2n+1)(n+1)}a_n$

$\underline{r=\frac{1}{2}}$: $\quad y_{1/2}(x)=\sqrt{x}\sum_{n=0}^{\infty}a_nx^n \qquad a_{n+1}=\dfrac{-1}{(2n+2)(n+3/2)}a_n$

Note: This time, choosing a_0 determines every a_n, but we still have 2 linearly independent sol'ns.

The general solution is $y(x)=A\,y_0(x)+B\,y_{1/2}(x)$, where y_0, $y_{1/2}$ are as above.

Summary: Consider, e.g., $2xy'' + y' + y = 0$.

(1) Write as $y'' + P(x)\,y' + Q(x)\,y = 0$ $\qquad P(x) = \frac{1}{2x}$, $Q(x) = \frac{1}{2x}$

$x = 0$ is a regular singular point $\qquad xP(x) = \frac{1}{2}$, $x^2 Q(x) = \frac{1}{2}x$.

(2) Assume $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$

(3) Plug back in, factor out x^r:

$$\cancel{x^r}\left[\sum_{n=0}^{\infty} (2n+2r-1)(n+r)\,a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n\right] = 0$$

(4) shift indices to get indicial eq'n.

$$\underbrace{(2r-1)\,r\,a_0}_{=0} x^{-1} + \sum_{n=0}^{\infty} \left[(2n+2r+1)(n+r)\,a_{n+1} + a_n\right] x^n = 0.$$

(5) Set coefficients $= 0$. Solve for r, & get recurrence relation.

Basic linear algebra:

Def: A vector space is a set (of vectors) X with a set of scalars (usually $\mathbb{R}$ or $\mathbb{C}$), that is

(i) Closed under addition: If $x_1, x_2 \in X$ then $x_1 + x_2 \in X$

(ii) Closed under scalar multiplication: If $x \in X$, then $cx \in X$ for any scalar c.

Ex: $\mathbb{R}^n$ is a vector space.

* closed under addition: $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1+y_1, \ldots, x_n+y_n) \in \mathbb{R}^n$. ✓
* closed under scalar mult: $c(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n) \in \mathbb{R}^n$. ✓

Ex: Let Poly_n = set of polynomials of degree $\leq n$.

This is a vector space, since $f(x) + g(x)$ and $c \cdot f(x)$ are degree $\leq n$ polynomials as long as f & g are.

Ex: Let PS = set of power series

* Closed under addition: $\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n$. ✓
* Closed under scalar mult.: $c \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} c a_n x^n$.

This is a vector space.

Ex: Let Per_T = set of T-periodic functions (i.e., $f(x+T) + f(x)$).

* Closed under addition: Say $f(x+T) = f(x)$ and $g(x+T) = g(x)$.

  Then $(f+g)(x+T) = f(x+T) + g(x+T) = f(x) + g(x) = (f+g)(x)$ ✓
* Closed under scalar mult: If $f(x+T) = f(x)$, then $c f(x+T) = c f(x)$ ✓.

Non-examples:

- Let X = unit circle in $\mathbb{R}^2$: $(1,0) + (1,0) \notin X$.
- Let X = upper half-plane in $\mathbb{R}^2$: $-5 \cdot (1,1) = (-5,-5) \notin X$.
- Let X = degree-n polynomials: $(x^n + 2) + (3x^2 - x^n) = 3x^2 + 2 \notin X$.

Def: If X is a vector space, then a basis is a (minimal) set of vectors $\{x_1, \ldots, x_n\}$ such that every vector $x \in X$ can be expressed uniquely as $x = c_1 x_1 + \cdots + c_n x_n$.

Ex: Let $X = \mathbb{R}^3$. $\mathcal{B} = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis,

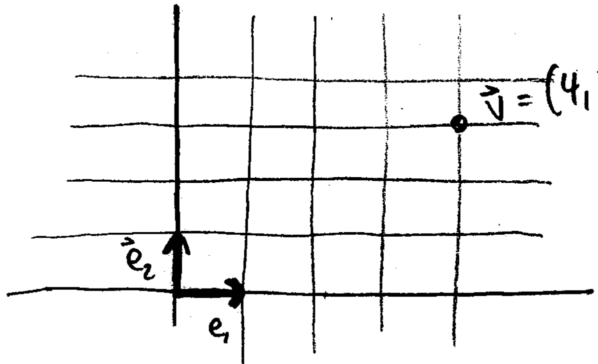
(call these: $\vec{e}_1$ $\vec{e}_2$ $\vec{e}_3$)

because $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$$= a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$$

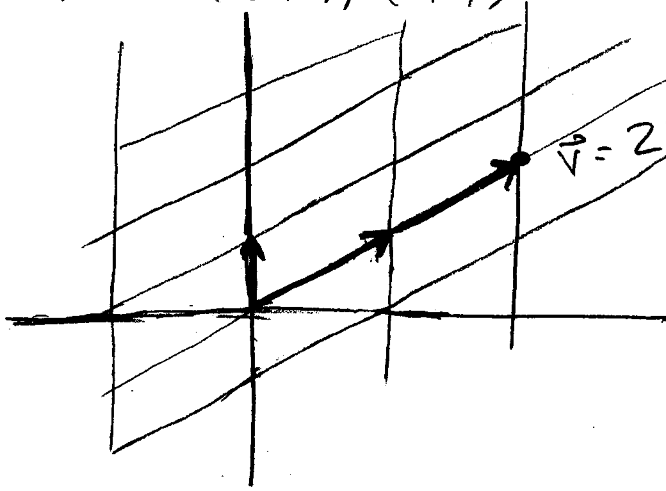
Ex: Let $X = \mathbb{R}^2$. $\mathcal{B} = \{ \overset{\vec{e}_1}{(1,0)}, \overset{\vec{e}_2}{(0,1)} \}$ is a basis.

Geometrically:



$\vec{v} = (4,3) = 4\vec{e}_1 + 3\vec{e}_2$

Note: $\{v_1, v_2\} = \{(2,1), (0,1)\}$ is also a basis.



$\vec{v} = 2\vec{v}_1 + 1\vec{v}_2 = (2,1)$ wrt this basis.

Note: In $\mathbb{R}^2$, $\{\vec{v}_1, v_2\}$ are a basis $\iff$ $\vec{v}_1 \neq c\vec{v}_2$.

Ex: Let Poly_n = set of polynomials of degree $\leq n$.

Then $\{1, x, x^2, \ldots, x^n\}$ is a basis.

Why? $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$, uniquely!

Note: $\{1, 2x, x^2, x^3, \ldots, x^{n-1}, x^n - 1\}$ is also a basis (why?)

Ex: Let PS = set of power series.

Then $\{1, x, x^2, x^3, \ldots\}$ is a basis, because $y(x) = \sum_{n=0}^{\infty} a_n x^n$, uniquely.

Important note: In some vector spaces, we "allow infinite sums," (e.g., power series), and in others, we don't (e.g, $\mathbb{R}^n$, Poly_n). It should be clear from the context when we do.

Fact: The set of solutions to an n^{th} order linear homog. ODE is a vector space.

Ex: $y''' - 7y' + 6y = 0$. Let $y(x) = e^{rt}$. $r = 1, 2, -3$, so

$y_1(t) = e^t$; $y_2(t) = e^{2t}$, $y_3(t) = e^{-3t}$.

$\{e^t, e^{2t}, e^{-3t}\}$ is a basis of the solution space (a vector space!)

i.e, every solution is of the form $y(t) = C_1 e^t + C_2 e^{2t} + C_3 e^{-3t}$, and solutions are closed under addition & scalar mult.

Note: $\{5e^t, e^{2t}, e^{-3t} + e^t\}$ is also a basis, but not as elegant of one.

Def: The dimension of a vector space is the size of its basis.

Ex: $\dim(\mathbb{R}^n) = n$ $\quad \mathcal{B} = \{\vec{e}_1, \ldots, \vec{e}_n\}$

$\dim(Poly_n) = n+1$ $\quad \mathcal{B} = \{1, x, x^2, \ldots, x^n\}$

$\dim(PS) = \infty$ $\quad \mathcal{B} = \{1, x, x^2, x^3, \ldots\}$

$\dim(Per_{2\pi}) = \infty$ $\quad \mathcal{B} = \{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\}$.

Def: A set of vectors $\{x_1, \ldots, x_m\}$ spans X if every vector $x = c_1 x_1 + \ldots + c_m x_m$.

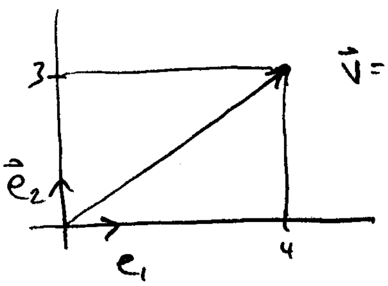
Ex: $\{\overset{\vec{v}_1}{(1,0)}, \overset{\vec{v}_2}{(0,1)}, \overset{\vec{v}_3}{(1,1)}\}$ spans $\mathbb{R}^2$, but isn't a basis.

It's not mimimal, and also, $(1,1) = 1\vec{v}_1 + 1\vec{v}_2 = 1\vec{v}_3$ (not unique!)

* Fundamental theorem of ODEs: The set of solutions to an n^{th} order linear homogeneous ODE is an n-dimensional vector space.

i.e, the general solution is $y(x) = C_1 y_1(x) + \ldots + C_n y_n(x)$.

Let's revisit basic geometry



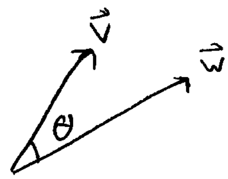
Question: How long is $\vec{v}$ in the x-direction?

Ans: 4 (duh).

Because $\vec{v} \cdot \vec{e}_1 = (4,3) \cdot (1,0) = 4.$

* Big idea: The dot product allows us to define angles, and hence distances, between vectors.

Fact: $\cos\theta := \dfrac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}$



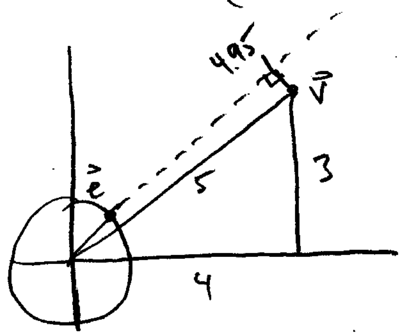
where $\|\vec{v}\| = \sqrt{|\vec{v} \cdot \vec{v}|} = \sqrt{v_1^2 + v_2^2 \cdots + v_n^2}$

This is the proper way to define angles.

* If $\|\vec{e}\| = 1$, then $\vec{v} \cdot \vec{e}$ = "length of $\vec{v}$ in the $\vec{e}$-direction"

= projection of $\vec{v}$ onto $\vec{e}$.

Ex: Let $\vec{e} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\vec{v} = (4,3)$



Question: How long is $\vec{v}$ in the NE-direction?

Ans: $\vec{v} \cdot \vec{e} = (4,3) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 7\frac{\sqrt{2}}{2} \approx 4.95$

Note: This works because $\vec{e}$ is a unit vector.

* Goal: We want to put a "dot product" on the space of periodic functions (fixed period T), so we can use these "geometric tools" to analyze them, and decompose them (into sines & cosines)