Week II Summary:

- Solving ODE's with power series & generalized power series.

  Singular points: Consider \( y'' + P(x)y' + Q(x)y = 0 \).
  
  * \( x_0 \) is ordinary if "\( P(x_0) \) and \( Q(x_0) \) are defined" (technically, "real analytic").
  * \( x_0 \) is singular, otherwise. In this case,
    - \( x_0 \) is regular if \((x-x_0)P(x)\) and \((x-x_0)^2Q(x)\) are real analytic.
    - \( x_0 \) is irregular, otherwise.

  **Theorem of Frobenius:**
  
  * If \( x_0 \) is an ordinary point, then there is a power series solution
    \[ y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n. \]
  
  * If \( x_0 \) is a regular singular point, then there is a generalized
    power series solution
    \[ y(x) = (x-x_0)^{\nu} \sum_{n=0}^{\infty} a_n (x-x_0)^n. \]
    Moreover, the radius of convergence is \( R = \min \{ R_0, R_0 \} \).

- Basic linear algebra:
  
  * A vector space \( X \) is a set closed under addition & scalar mul.
  * A basis for \( X \) is a min'l set of vectors that spans \( X \).
  * The dot product allows us to project vectors onto unit vectors,
    and measure lengths & angles.
Let's extend the notion of a dot product in \( \mathbb{R}^n \) to an arbitrary vector space (we call it an "inner product").

**Def:** Let \( X \) be a vector space. An inner product is a function (denoted, e.g., \( \langle u, v \rangle \)) such that:

(i) \( \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \) (additive)

(ii) \( \langle cu, v \rangle = c \langle u, v \rangle \) (constants pull out)

(iii) \( \langle v, w \rangle = \langle w, v \rangle \) (symmetric)

(iv) \( \langle v, v \rangle \geq 0 \) (non-negative)

(v) \( \langle v, v \rangle = 0 \) iff \( v = 0 \)

**Def:** If \( \langle v, w \rangle = 0 \), then \( v \) \& \( w \) are **orthogonal** (perpendicular).

- A set of vectors \( \{v_1, \ldots, v_n\} \) is **orthonormal** if they are all unit-length and mutually orthogonal (i.e., \( v_i \cdot v_j = 0 \) \( i \neq j \)).

*Big Idea:* Orthonormal bases are really nice!

**Ex:** Consider \( \mathbb{R}^2 \), let \( \vec{v} = (4, 3, 2) \),

\[
\vec{v} = (4, 3, 2) = (\vec{v} \cdot \vec{e}_1, \; \vec{v} \cdot \vec{e}_2, \; \vec{v} \cdot \vec{e}_3)
\]

This works because with the dot product, \( \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \) is an orthonormal basis for \( \mathbb{R}^3 \):

- Unit length: \( \|e_i\| = 1 \), i.e., \( \langle e_i, e_i \rangle = 1 \)
- Orthogonality (perpendicular): \( \langle e_i, e_j \rangle = 0 \) if \( i \neq j \)

\[
\vec{v} = (4, 3) = 4\vec{e}_1 + 3\vec{e}_2
\]
Now, we'll do the same thing in $\text{Per}_{2\pi}$, the space of $2\pi$-periodic functions.

Define an inner product on $\text{Per}_{2\pi}$ as follows:

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) g(t) \, dt$$

**Fact 1:** $B = \{ \frac{1}{2}, \cos t, \cos 2t, \ldots, \sin t, \sin 2t, \ldots \}$ is a basis of $\text{Per}_{2\pi}$.

**Fact 2:** This inner product makes $B$ an orthonormal basis of $\text{Per}_{2\pi}$!

i.e.,

$$\langle \cos nt, \cos mt \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nt \cos mt \, dt = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases},$$

$$\langle \sin nt, \sin mt \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nt \sin mt \, dt = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}.$$

$$\langle \cos nt, \sin mt \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nt \sin mt \, dt = 0.$$

**Big idea:** Since $B$ is a basis, every $2\pi$-periodic function can be written as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt.$$

Since $B$ is orthonormal, we can use our inner product to decompose vectors into components by projection.

**Compare:**

In $\mathbb{R}^2$, $(4,3) \cdot (1,0) = 4$ "magnitude in the x-direction"

In $\text{Per}_{2\pi}$, $\langle f, \cos 2t \rangle = a_2$ "magnitude in the $\cos 2t$-direction"

Thus, we have a formula for the coefficients $a_n$ and $b_n$:

$$a_n = \langle f(t), \cos nt \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt,$$

$$b_n = \langle f(t), \sin nt \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt.$$
\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n \pi x}{L} + b_n \sin \frac{n \pi x}{L} \]

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n \pi x}{L} \, dx \]

\[ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n \pi x}{L} \, dx \]

[Show Demo: www.falstad.com/fourier]

**Example:** Square wave:

\[ f(t) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nt + b_n \sin nt \]

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, dt = \frac{1}{\pi} \int_{-\pi}^{0} 1 \, dt + \frac{1}{\pi} \int_{0}^{\pi} 1 \, dt = 0. \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt = \frac{1}{\pi} \int_{-\pi}^{0} -1 \cos nt \, dt + \frac{1}{\pi} \int_{0}^{\pi} 1 \cos nt \, dt \]

\[ = \frac{1}{\pi} \left. \sin nt \right|_{-\pi}^{0} + \frac{1}{\pi} \left. \sin nt \right|_{0}^{\pi} = 0. \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt = \frac{1}{\pi} \int_{-\pi}^{0} -1 \sin nt \, dt + \frac{1}{\pi} \int_{0}^{\pi} 1 \sin nt \, dt \]

\[ = \frac{1}{\pi} \left. \cos nt \right|_{-\pi}^{0} - \frac{1}{\pi} \left. \cos nt \right|_{0}^{\pi} = \frac{1}{\pi} (1 - \cos n \pi) - \frac{1}{\pi} (\cos n \pi - 1) \]

\[ = \frac{2}{\pi} (1 - \cos n \pi). \]

\[ a_n = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n \pi} & n \text{ odd} \end{cases} \]

Note: \( \cos n \pi = (-1)^n \)
\[ f(t) = \frac{4}{\pi} \sin t + \frac{4}{3 \pi} \sin 3t + \frac{4}{5 \pi} \sin 5t + \frac{4}{7 \pi} \sin 7t + \ldots \]

Note: All cosine terms, i.e., even sine terms, are zero. (Why?)

\[ \begin{array}{c}
\cos t \\
\sin 2t \\
\sin t \\
distinct waves
\end{array} \]

This "looks more like a sine than a cosine wave."

**Ex 2:** Sawtooth wave

\[ f(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } 0 \leq t \leq \pi \\ \pi & \text{for } t > \pi \end{cases} \]

\[ f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt. \]

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} t \, dt = 0. \quad \text{(By symmetry)} \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t \cos nt \, dt \]

\[ = \frac{1}{\pi} \left[ \frac{1}{n} t \sin nt \bigg|_{-\pi}^{\pi} - \frac{1}{n} \int_{-\pi}^{\pi} \sin nt \, dt \right] \]

\[ = \frac{1}{n \pi} \int_{-\pi}^{\pi} \sin nt = \frac{1}{n \pi} \cos nt \bigg|_{-\pi}^{\pi} = \frac{1}{n^2 \pi} \left[ \cos n\pi t - \cos(-n\pi t) \right] \]

\[ = \frac{1}{n^2 \pi} \left[ \cos n\pi t - \cos n\pi t \right] = 0. \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t \sin nt \, dt \]

\[ = \frac{1}{\pi} \left[ \frac{-1}{n} t \cos nt \bigg|_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} \cos nt \, dt \right] \]

\[ = \frac{1}{\pi n} \left[ (\frac{-\pi}{n} \cos n\pi) - (\frac{\pi}{n} \cos n\pi) + \frac{1}{n^2} \sin nt \bigg|_{-\pi}^{\pi} \right] \]
\[
= \frac{1}{n} \left[ -\frac{2\pi}{n} \cos(n\pi) \right] = -\frac{2}{n} \cos(n\pi) = -\frac{2}{n} (-1)^n = \frac{2}{n} (-1)^{n+1} = \begin{cases} 
-\frac{2}{n} & \text{if } n \text{ even} \\
\frac{2}{n} & \text{if } n \text{ odd} 
\end{cases}
\]

Thus, \( f(t) = 2 \sin t - \frac{2}{3} \sin 3t + \frac{2}{5} \sin 5t + \ldots \)

\[= 2 \sin t - \frac{2}{3} \sin 3t + \frac{2}{5} \sin 5t + \ldots \]

Think: How does this relate to music, sound waves, or harmonics?

**Exploiting Symmetry**

Why are many of the \( a_n \) 's and \( b_n \) 's zero?

**Def:**
- \( f(t) \) is an **even function** if \( f(t) = f(-t) \)
- \( f(t) \) is an **odd function** if \( f(t) = -f(-t) \).

**Graphically:**

- \( f(t) \) even \( \iff \) symmetric about the \( y \)-axis.
- \( f(t) \) odd \( \iff \) symmetric about the origin.

**Why we care:**
- If \( f(t) \) is even, then \( \int_{-L}^{L} f(t) \, dt = 2 \int_{0}^{L} f(t) \, dt \) (look at the area under the curve to see why!)
- If \( f(t) \) is odd, then \( \int_{-L}^{L} f(t) \, dt = 0 \)
Facts:
- If $f$ and $g$ are even, then $f(t)g(t)$ is even.
- If $f$ and $g$ are odd, then $f(t)g(t)$ is even.
- If $f$ is even and $g$ is odd, then $f(t)g(t)$ is odd.

Examples:
- Even Functions: $8$, $t^2$, $3t^2 + t^2 - 5$, $|t|$,
  
  $\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots = \frac{e^{it} + e^{-it}}{2}$

  $\cosh t = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots = \frac{e^t + e^{-t}}{2}$

- Odd Functions: $2t$, $8t^3 - 5t$,
  
  $\sin t = 1 - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots = \frac{e^{it} - e^{-it}}{2}$

  $\sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \cdots = \frac{e^t - e^{-t}}{2}$

- Neither: $t^2 - 3t + 2$, $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots = \cosh t + \sinh t$.

Note:
- If $F(t)$ is even, then $F(t)\cos nt$ is even $\Rightarrow a_n = \frac{2}{L}\int_0^L F(t)\cos\left(\frac{nt\pi}{L}\right)dt$
  
  and $F(t)\sin nt$ is odd $\Rightarrow b_n = 0$ (all $n$)

- If $F(t)$ is odd, then $F(t)\cos nt$ is odd $\Rightarrow a_n = 0$ (all $n$)
  
  and $F(t)\sin nt$ is even $\Rightarrow b_n = \frac{2}{L}\int_0^L F(t)\sin\left(\frac{nt\pi}{L}\right)dt$
Fourier sine & cosine series

Idea: Consider some function defined on \([0, L]\)

Find "the Fourier series of \(f(t)\)."

First, we need to make \(f(t)\) periodic.

A naïve extension

The even extension of \(f(t)\)

The odd extension of \(f(t)\)

- Fourier series of the even extension:
  \[
  a_n = \frac{2}{L} \int_0^L f(t) \cos \left( \frac{n \pi t}{L} \right) \, dt
  \]
  \[
  b_n = 0
  \]
  (called the Fourier cosine series of \(f(t)\))

- Fourier series of the odd extension:
  \[
  a_n = 0
  \]
  \[
  b_n = \frac{2}{L} \int_0^L f(t) \sin \left( \frac{n \pi t}{L} \right) \, dt
  \]

Example: \(f(t) = t\) on \([0, L]\)

Compute the Fourier sine & cosine series.

Odd extension:

\[ f(t) \]

Fourier sine series:

\[
 f(t) = \sum_{n=1}^{\infty} b_n \sin nt
\]

\[
 b_n = \begin{cases} 
 -\frac{2}{n\pi} & n \text{ even} \\
 \frac{2}{n\pi} & n \text{ odd} 
\end{cases}
\]

(we already did this).
Even extension

**Fourier cosine series**: \( f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt \)

\[ a_0 = \frac{2}{\pi} \int_{0}^{\pi} t \cos 0 \, dt = \frac{t^2}{\pi} \bigg|_{0}^{\pi} = \pi \]

\[ a_n = \frac{2}{\pi} \int_{0}^{\pi} t \cos nt \, dt = \frac{2}{\pi} \left[ \frac{t}{n} \sin nt \bigg|_{0}^{\pi} - \frac{1}{n} \int_{0}^{\pi} \sin nt \, dt \right] \]

Let \( u = t \) \quad \( v = \frac{1}{n} \sin nt \) \quad \( du = dt \) \quad \( dv = \cos nt \, dt \)

\[ = \frac{2}{\pi n^2} \cos nt \bigg|_{0}^{\pi} = \frac{2}{\pi n^2} \left[ \cos n\pi - 1 \right] \]

\[ = \frac{2}{\pi n^2} \left[ (-1)^n - 1 \right] = \begin{cases} 0 & n \text{ even} \\ \frac{-4}{\pi n^2} & n \text{ odd} \end{cases} \]

\[ f(t) = \frac{\pi}{2} - \frac{4}{\pi} \cos t - \frac{4}{9\pi} \cos 3t - \frac{4}{25\pi} \cos 5t - \frac{4}{49\pi} \cos 7t - \ldots \]

**Example**: \( f(t) = \begin{cases} t & 0 \leq t < \frac{\pi}{2} \\ \pi - t & \frac{\pi}{2} \leq t < \pi \end{cases} \)

Compute the Fourier sine series.

Observe the symmetry about the line \( t = \frac{\pi}{2} \):

\( \sin t \) \quad \( \sin 2t \) \quad \( \sin 3t \) \quad \( \sin 4t \)

\( \sin nt \) has "even symmetry about \( t = \frac{\pi}{2} \)" if \( n \) is odd,\nand "odd symmetry about \( t = \frac{\pi}{2} \)" if \( n \) is even.
Since \( f(t) \) has "even symmetry about \( t = \frac{\pi}{2} \), \( b_n = 0 \) for all even \( n \),
and when \( n \) is odd, \( f(t) \sin nt \) has even symmetry about \( t = \frac{\pi}{2} \),
i.e., \( b_n = \frac{2}{\pi} \int_{0}^{\pi/2} f(t) \sin nt \; dt = \frac{4}{\pi} \int_{0}^{\pi/2} f(t) \sin nt \; dt \)

\[
= \frac{4}{\pi} \int_{0}^{\pi/2} t \sin nt \; dt = \frac{4}{\pi} \left[ \frac{t}{n} \cos nt \right]_{0}^{\pi/2} + \int_{0}^{\pi/2} \frac{1}{n} \cos nt \; dt \\
= \frac{4}{\pi} \left[ \frac{\pi}{2n} \cos \left( \frac{\pi n}{2} \right) - 0 + \frac{1}{n^2} \sin nt \right]_{0}^{\pi/2} \quad \text{(since \( n \) is odd)} \\
= \frac{4}{\pi} \left[ \frac{1}{n^2} \sin \left( \frac{\pi n}{2} \right) \right] \quad \text{Note:} \quad \sin \frac{\pi n}{2} = \begin{cases} 0 & n = 4k \\ 1 & n = 4k+1 \\ 0 & n = 4k+2 \\ -1 & n = 4k+3 \end{cases}
\]

Thus, \( b_n = \begin{cases} \\
0 & n = 4k \\
\frac{4}{n^2} \pi & n = 4k+1 \\
0 & n = 4k+2 \\
-\frac{4}{n^2} \pi & n = 4k+3 \\
\end{cases} \)

So, \( f(t) = \frac{4}{\pi} \sin t - \frac{4}{9\pi} \sin 3t + \frac{4}{25\pi} \sin 5t - \frac{4}{49\pi} \sin 7t + \cdots \)

**Complex form of the Fourier series**

**Fact 1:** \( \mathcal{B}_1 = \left\{ \frac{1}{\sqrt{2}}, \cos t, \cos 2t, \cos 3t, \ldots \right\} \) is a basis for \( \operatorname{Per}_{2\pi}, \)

and is orthonormal if \( \langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) g(t) \; dt \)

**Fact 2:** \( \mathcal{B}_2 = \left\{ 1, e^{-it}, e^{-2it}, e^{-3it}, \ldots \right\} \) is also a basis,

and is orthonormal if \( \langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) g(t) \; dt \).

Therefore, if \( f(t) \) is \( 2\pi \)-periodic, then

\[
f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} = c_0 + \sum_{n=1}^{\infty} \left( c_n e^{int} + c_{-n} e^{-int} \right) = c_0 + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \; dt,
\]

\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{int} \; dt,
\]
This is the complex form of the Fourier series of f(t).

Recall: \[\cos nt = \frac{1}{2}(e^{int} + e^{-int}), \quad \sin nt = \frac{1}{2i}(e^{int} - e^{-int})\]

\[e^{int} = \cos nt + i \sin nt\]

Therefore, \[C_n = \frac{a_n - ib_n}{2}, \quad C_{-n} = \frac{a_n + ib_n}{2}\]

and \[a_n = C_n + C_{-n}, \quad b_n = i(C_n - C_{-n})\]

Note: \(C_0\) is the constant term in the complex form of \(f(t)\).

\(a_0 = 2C_0 \Rightarrow \frac{a_0}{2}\) is the constant term in the real form.