MTHSC 208, Week 12

Week 11 summary:

- Solving ODEs with power series & generalized power series.

Singular points: Consider $y'' + P(x)y' + Q(x)y = 0$.

* x_0 is ordinary if "$P(x_0)$ and $Q(x_0)$ are defined" (technically, "real analytic")

* x_0 is singular, otherwise. In this case,

- x_0 is regular if $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ are real analytic.
- x_0 is irregular, otherwise.

Theorem of Frobenius:

* If x_0 is an ordinary point, then there is a power series sol'n
$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$.

* If x_0 is a regular singular point, then there is a generalized power series sol'n $y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n$.

Moreover, the radius of convergence is $R = \min\{R_P, R_Q\}$.

- Basic linear algebra:

* A vector space X is a set closed under addition & scalar mult.

* A basis for X is a min'l set of vectors that spans X.

* The dot product allows us to project vectors onto unit vectors, and measure lengths & angles.

Let's extend the notion of a dot product in $\mathbb{R}^n$ to an arbitrary vector space (we call it an "inner product").

<u>Def:</u> Let X be a vector space. An <u>inner product</u> is a function (denoted, e.g., $\langle u, v\rangle$ such that):

(i) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ (additive)

(ii) $\langle cu, v\rangle = c\langle u, v\rangle$ (constants pull out)

(iii) $\langle v, w\rangle = \langle w, v\rangle$ (symmetric)

(iv) $\langle v, v\rangle \geq 0$ (non-negative)

(v) $\langle v, v\rangle = 0$ iff $v=0$

<u>Def:</u>
- If $\langle v, w\rangle = 0$, then v & w are <u>orthogonal</u> (perpendicular).
- A set of vectors $\{v_1, \dots, v_n\}$ is <u>orthonormal</u> if they are all unit-length and mutually orthogonal (i.e, $v_i \cdot v_j = 0$ $i \neq j$).

* <u>Big idea:</u> Orthonormal bases are really nice!

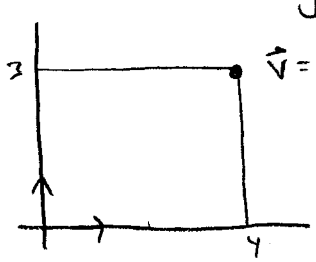
<u>Ex:</u> Consider $\mathbb{R}^3$, let $\vec{v} = (4,3,2)$.

$\vec{v} = (4,3,2) = (\vec{v}\cdot\vec{e}_1, \vec{v}\cdot\vec{e}_2, \vec{v}\cdot\vec{e}_3)$.

This works because with the dot product, $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is an orthonormal basis for $\mathbb{R}^3$:

* Unit length: $\|e_i\| = 1$, i.e, $\boxed{\langle e_i, e_i\rangle = 1}$

* Orthogonality (perpendicular): $\boxed{\langle e_i, e_j\rangle = 0}$ if $i \neq j$



$\vec{v} = (4,3) = 4\vec{e}_1 + 3\vec{e}_2$

Now, we'll do the same thing in $Per_{2\pi}$, the space of 2π-periodic functions.

Define an inner product on $Per_{2\pi}$ as follows:

$$\boxed{\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, g(t)\, dt}$$

Fact 1: $\mathcal{B} = \{ \frac{1}{2}, \cos t, \cos 2t, \ldots$
$\sin t, \sin 2t, \ldots \}$ is a basis of $Per_{2\pi}$.

Fact 2: This inner product makes $\mathcal{B}$ an orthonormal basis of $Per_{2\pi}$!

i.e, $\langle \cos nt, \cos mt \rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nt \cos mt \, dt = \begin{cases} 1 & n = m \\ 0 & n \neq m, \end{cases}$

$\langle \sin nt, \sin mt \rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nt \sin mt \, dt = \begin{cases} 1 & n = m \\ 0 & n \neq m, \end{cases}$

$\langle \cos nt, \sin mt \rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nt \sin mt \, dt = 0.$

* Big idea: Since $\mathcal{B}$ is a basis, every 2π-periodic function can be written as $\boxed{f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt}$

* Since $\mathcal{B}$ is orthonormal, we can use our inner product to decompose vectors into components by projection:

Compare: In $\mathbb{R}^2$, $(4,3)\cdot(1,0) = 4$ "magnitude in the x-direction"

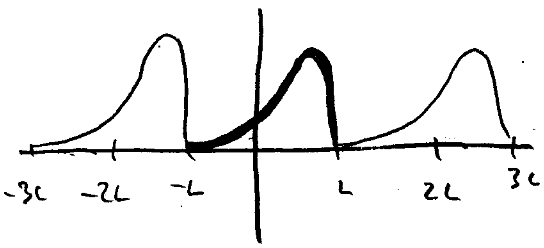
In $Per_{2\pi}$, $\langle f, \cos 2t \rangle = a_2$ "magnitude in the $\cos 2t$-direction"

Thus, we have a formula for the coefficients a_n & b_n:

$$\boxed{\begin{aligned} a_n &= \langle f(t), \cos nt \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt \, dt \\ b_n &= \langle f(t), \sin nt \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt \, dt \end{aligned}}$$

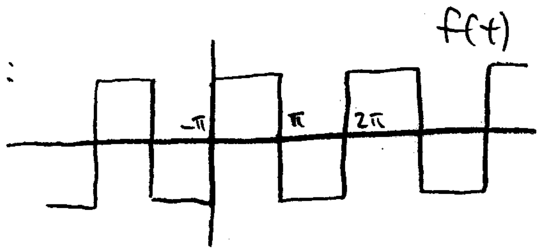
Note: This easily generalizes to functions of period $2L$ (not just 2π)

$$\boxed{\begin{aligned} * : f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{aligned}}$$



[SHOW DEMO: www.falstad.com/fourier]

Ex 1: Square wave:



Find the Fourier series of $f(t)$:

$$f(t) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nt + b_n \sin nt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} -1\,dt + \frac{1}{\pi}\int_0^{\pi} 1\,dt = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_{-\pi}^{0} -1\cos nt\,dt + \frac{1}{\pi}\int_0^{\pi} 1\cos nt\,dt$$

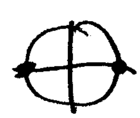
$$= \frac{-1}{n\pi}\sin nt\Big|_{-\pi}^{0} + \frac{1}{n\pi}\sin nt\Big|_0^{\pi} = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt = \frac{1}{\pi}\int_{-\pi}^{0} -1\sin nt\,dt + \frac{1}{\pi}\int_0^{\pi} 1\sin nt\,dt$$

$$= \frac{1}{n\pi}\cos nt\Big|_{-\pi}^{0} - \frac{1}{n\pi}\cos nt\Big|_0^{\pi} = \frac{1}{n\pi}(1-\cos n\pi) - \frac{1}{n\pi}(\cos n\pi - 1)$$

$$= \frac{2}{n\pi}(1-\cos n\pi).$$

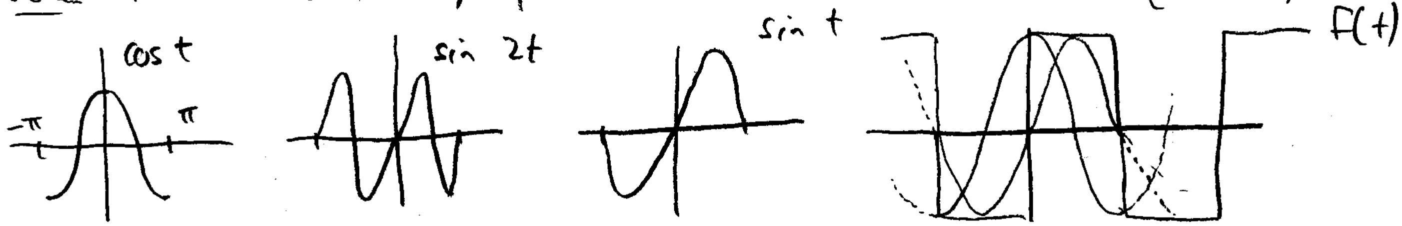
Note: $\cos n\pi = (-1)^n$



$$\Rightarrow \boxed{b_n = \frac{2}{n\pi}(1-\cos n\pi) = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}}$$

i.e., $\boxed{f(t) = \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\sin 5t + \frac{4}{7\pi}\sin 7\pi\, t \ldots}$

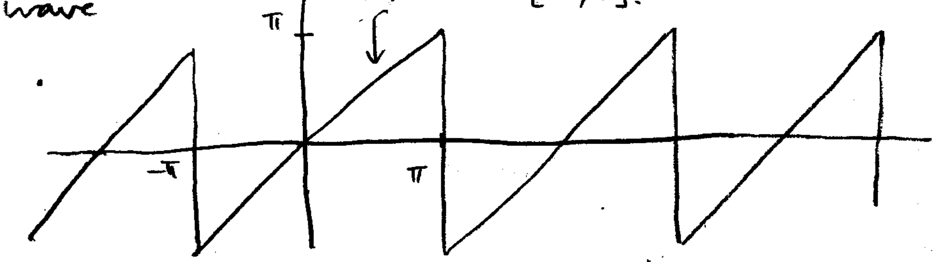
Note: All cosine terms, & "even" sine terms, are zero. (Why?)



This "looks more like a sine than a cosine wave."

Ex 2: Sawtooth wave

$f(t) = t$ on $[-\pi, \pi]$.



$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t\, dt = 0. \quad \text{(By symmetry)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t \cos nt\, dt \qquad \text{let } u = t \quad v = \frac{1}{n}\sin nt$$
$$du = dt \quad dv = \cos nt\, dt$$

$$= \frac{1}{\pi}\left[\frac{1}{n} t \overset{0}{\cancel{\sin}} nt \Big|_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi} \sin nt\, dt\right]$$

$$= \frac{-1}{n\pi}\int_{-\pi}^{\pi} \sin nt = \frac{1}{n^2\pi}\cos nt \Big|_{-\pi}^{\pi} = \frac{1}{n^2\pi}\left[\cos \pi t - \cos(-\pi t)\right]$$

$$= \frac{1}{n^2\pi}\left[\cos \pi t - \cos \pi t\right] = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t \sin nt\, dt \qquad \text{let } u = t \quad v = -\frac{1}{n}\cos nt$$
$$du = dt \quad dv = \sin nt\, dt$$

$$= \frac{1}{\pi}\left[-\frac{1}{n} t \cos nt \Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} \cos nt\, dt\right]$$

$$= \frac{1}{\pi n}\left[\left(-\frac{\pi}{n}\cos n\pi\right) - \left(\frac{\pi}{n}\cos n\pi\right) + \frac{1}{n^2}\overset{0}{\cancel{\sin}} nt \Big|_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos(n\pi)\right] = -\frac{2}{n}\cos n\pi = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1} = \begin{cases} -2/n & n \text{ even} \\ 2/n & n \text{ odd} \end{cases}$$

Thus, $f(t) = 2\sin t - \frac{2}{2}\sin 2t + \frac{2}{3}\sin 3t - \frac{2}{4}\sin 4t + \frac{2}{5}\sin 5t \ +/- \ldots$

$$= 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \frac{2}{5}\sin 5t \ +/- \ldots$$

Think: How does this relate to music, sound waves, & harmonics?

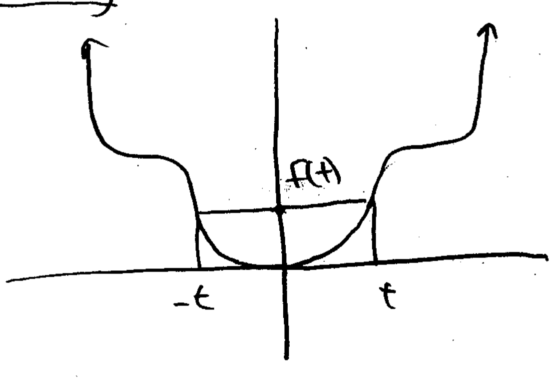
Exploiting Symmetry

Why are many of the a_n's and b_n's zero?

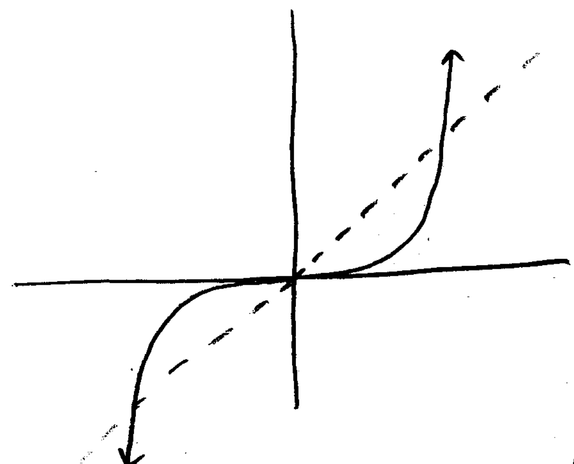
Def: • $f(t)$ is an even function if $f(t) = f(-t)$

• $f(t)$ is an odd function if $f(t) = -f(-t)$.

Graphically:



$f(t)$ even $\Leftrightarrow$ symmetric about the y-axis.



$f(t)$ odd $\Leftrightarrow$ symmetric about the origin

Why we care:

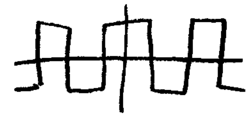
• If $f(t)$ is even, then $\int_{-L}^{L} f(t)\,dt = 2\int_0^L f(t)\,dt$

• If $f(t)$ is odd, then $\int_{-L}^{L} f(t)\,dt = 0$

Look at the area under the curve to see why!

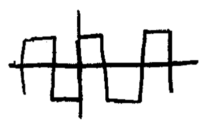
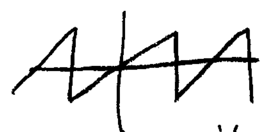
Facts:
- If f & g are even, then $f(t)\,g(t)$ is even
- If f & g are odd, then $f(t)\,g(t)$ is even
- If f is even & g is odd, then $f(t)\,g(t)$ is odd.

Examples:

- Even Functions: 8, t^2, $3t^6+t^2-5$, $|t|$ 

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} +/- \cdots = \frac{e^{it}+e^{-it}}{2}$$

$$\cosh t = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots = \frac{e^{t}+e^{-t}}{2}$$

- Odd Functions: $2t$, $8t^3-5t$,  

$$\sin t = 1 - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} +/- \cdots = \frac{e^{it}-e^{-it}}{2i}$$

$$\sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \cdots = \frac{e^{t}-e^{-t}}{2}$$

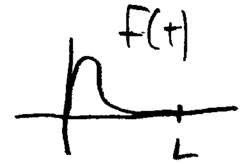
- Neither: t^2-3t+2, $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots = \cosh t + \sinh t.$

Note:
- If $f(t)$ is even, then $f(t)\cos nt$ is even $\Rightarrow a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt$

  and $f(t)\sin nt$ is odd $\Rightarrow b_n = 0$ (all n)
- If $f(t)$ is odd, then $f(t)\cos nt$ is odd $\Rightarrow a_n = 0$ (all n)

  and $f(t)\sin nt$ is even $\Rightarrow b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt$

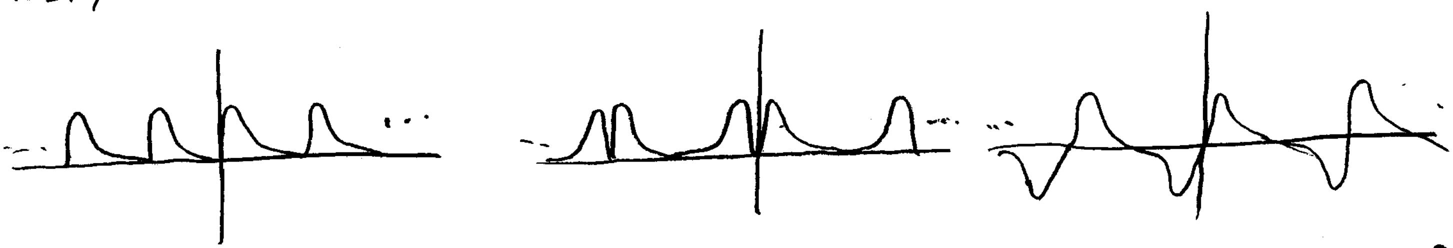
Fourier sine & cosine series



Idea: Consider some function defined on $[0, L]$

Find "the Fourier series of $f(t)$."

First, we need to make $f(t)$ periodic.



A naive extension — The even extension of $f(t)$ — The odd extension of $f(t)$

- Fourier series of the even extension: $\begin{cases} a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt \\ b_n = 0 \end{cases}$

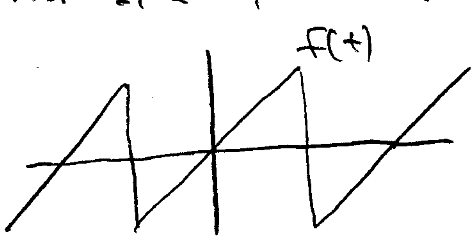
(called the Fourier cosine series of $f(t)$)

- Fourier series of the odd extension: $\begin{cases} a_n = 0 \\ b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt \end{cases}$

Example: Let $f(t) = t$ on $[0, t]$

Compute the Fourier sine & cosine series.

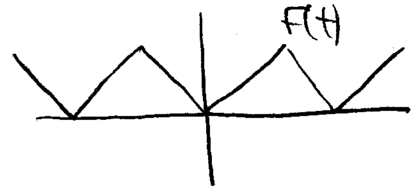
odd extension:



Fourier sine series: $f(t) = \sum_{n=1}^{\infty} b_n \sin nt$ $\qquad b_n = \begin{cases} -2/n\pi & n \text{ even} \\ 2/n\pi & n \text{ odd} \end{cases}$

(we already did this).

even extension



Fourier cosine series: $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} t\,dt = \frac{t^2}{\pi}\Big|_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} t\cos nt\,dt = \frac{2}{\pi}\left[\frac{t}{n}\cancel{\sin nt}\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin nt\,dt\right]$$

let $u = t$ $\quad v = \frac{1}{n}\sin nt$

$du = dt$ $\quad dv = \cos nt\,dt$

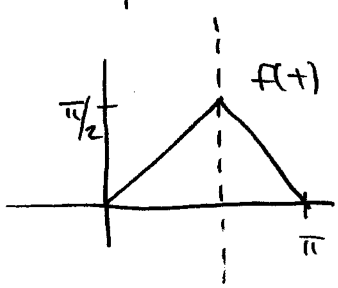
$$= \frac{2}{\pi n^2}\cos nt\Big|_0^{\pi} = \frac{2}{n^2\pi}\left[\cos n\pi - 1\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} 0 & n \text{ even} \\ \frac{-4}{\pi n^2} & n \text{ odd} \end{cases}$$

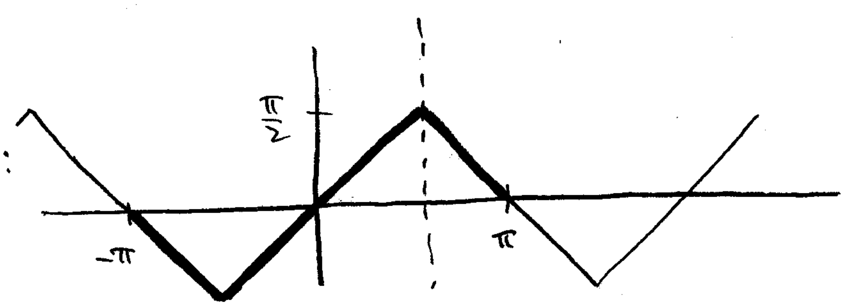
$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\cos t - \frac{4}{9\pi}\cos 3t - \frac{4}{25\pi}\cos 5t - \frac{4}{49\pi}\cos 7t - \dots$$

Example: $f(t) = \begin{cases} t & 0 \le t < \pi/2 \\ \pi - t & \pi/2 \le t < \pi \end{cases}$

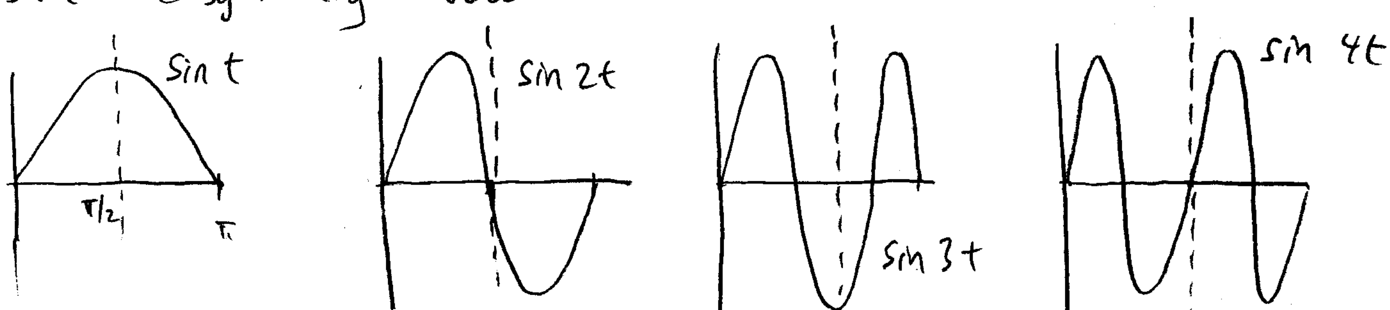
Compute the Fourier sine series.



odd extension:



Observe the symmetry about the line $t = \pi/2$:



* $\sin nt$ has "even symmetry about $t = \pi/2$" if n is odd,
and "odd symmetry about $t = \pi/2$" if n is even.

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Since $f(t)$ has "even symmetry about $t=\pi/2$," $b_n = 0$ for all even n.

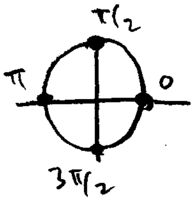
and when n is odd, $f(t)\sin nt$ has even symmetry about $t=\pi/2$,

i.e, $b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt = \frac{4}{\pi}\int_0^{\pi/2} f(t)\sin nt\,dt$

$$= \frac{4}{\pi}\int_0^{\pi/2} t\sin nt\,dt = \frac{4}{\pi}\left[\frac{t}{n}\cos nt\Big|_0^{\pi/2} + \int_0^{\pi/2}\frac{1}{n}\cos nt\,dt\right]$$

$$= \frac{4}{\pi}\left[\frac{\pi}{2n}\cancelto{0}{\cos\left(\frac{n\pi}{2}\right)} - 0 + \frac{1}{n^2}\sin nt\Big|_0^{\pi/2}\right] \quad \text{(since } n \text{ is odd)}$$

$$= \frac{4}{\pi}\left[\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]$$



Note: $\sin\frac{n\pi}{2} = \begin{cases} 0 & n=4k \\ 1 & n=4k+1 \\ 0 & n=4k+2 \\ -1 & n=4k+3 \end{cases}$

Thus, $b_n = \begin{cases} 0 & n=4k \\ 4/n^2\pi & n=4k+1 \\ 0 & n=4k+2 \\ -4/n^2\pi & n=4k+3 \end{cases}$

So, $f(t) = \frac{4}{\pi}\sin t - \frac{4}{9\pi}\sin 3t + \frac{4}{25\pi}\sin 5t - \frac{4}{49\pi}\sin 7t +\!/\!- \cdots$

Complex form of the Fourier series

Fact 1: $\mathcal{B}_1 = \left\{\begin{matrix}\frac{1}{2}, & \cos t, & \cos 2t, & \cos 3t, \ldots \\ & \sin t, & \sin 2t, & \sin 3t, \ldots\end{matrix}\right\}$ is a basis for $Per_{2\pi}$,

and is orthonormal if $\langle f,g\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt$

Fact 2: $\mathcal{B}_2 = \left\{\begin{matrix}1, & e^{-it}, & e^{-2it}, & e^{-3it}, \ldots \\ & e^{it}, & e^{2it}, & e^{3it}, \ldots\end{matrix}\right\}$ is also a basis,

and is orthonormal if $\langle f,g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt$.

Therefore, if $f(t)$ is 2π-periodic, then

$$\boxed{\begin{gathered} f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-int} = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{-int} + c_{-n}e^{int}\right) \\ c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt \end{gathered}}$$

This is the <u>complex form</u> of the Fourier series of $f(t)$.

<u>Recall</u>: $\cos nt = \frac{1}{2}(e^{it}+e^{-it})$, $\sin nt = \frac{1}{2i}(e^{it}-e^{-it})$

$$e^{int} = \cos nt + i \sin nt$$

Therefore, $\boxed{c_n = \dfrac{a_n - ib_n}{2}, \quad c_{-n} = \dfrac{a_n + ib_n}{2}}$

and $\boxed{a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n})}$

<u>Note</u>: c_0 is the const. term in the complex form of $f(t)$.

$a_0 = 2c_0 \Rightarrow \frac{a_0}{2}$ is the const. term in the real form.