Week 12 summary

- For a vector space $V$, we can define an inner product $\langle \cdot, \cdot \rangle$ (generalized dot product). This allows us to measure vectors (angles & lengths), and project vectors.

- For $\text{Per}_L$, define $\langle f, g \rangle = \frac{1}{L} \int_{-L}^{L} f(t) g(t) \, dt$

  + If $f(t)$ is $2\pi$-periodic, then

    $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt.$$ 

    $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt.$$

- Even Functions:
  - $f(t) = f(-t)$
  - Symmetric about $y$-axis
  - Fourier series contains only cosine terms
  - $\int_{-L}^{L} f(t) \, dt = 2 \int_{0}^{L} f(t) \, dt$.

- Odd Functions:
  - $f(t) = -f(-t)$
  - Symmetric about origin
  - Fourier series contains only sine terms
  - $\int_{-L}^{L} f(t) \, dt = 0$.

- Fourier cosine & sine series:

  * Start with a function $f(t)$ on $[0, L]$, e.g., $f(t)$
  * Fourier cosine series is the Fourier series of the even extension

  $f(t)$

  $$a_n = \frac{2}{L} \int_{0}^{L} f(t) \cos \left( \frac{n\pi t}{L} \right) \, dt$$
* Fourier sine series is the Fourier series of the odd extension.

\[ b_n = \frac{2}{L} \int_0^L f(t) \sin \left( \frac{n\pi t}{L} \right) \, dt. \]

* Real Fourier series: \( f(t) = \frac{a_0}{2} + \sum_{n=1}^\infty \left[ a_n \cos nt + b_n \sin nt \right] \)

Complex Fourier series: \( f(t) = c_0 + \sum_{n=1}^\infty \left( c_n e^{-int} + c_{-n} e^{int} \right) \)

\[ = \sum_{n=-\infty}^{\infty} c_n e^{-int} \]

Recall: \( e^{int} = \cos nt + i \sin nt, \quad \cos nt = \frac{e^{int} + e^{-int}}{2}, \quad \sin nt = \frac{e^{int} - e^{-int}}{2i} \)

Thus, \( a_0 = c_0 + c_{-0}, \quad b_n = i(c_n - c_{-n}), \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2} \)

* Key idea: \( B = \{ e^{-in\pi} : n \in \mathbb{Z} \} \) is a (better) basis for \( P_{2\pi} \).

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Complex Fourier series

Ex 1: Compute the complex Fourier series of

\[ f(t) = \begin{cases} \frac{1}{2} & -\pi \leq t < 0 \\ \frac{1}{2} & 0 < t < \pi \end{cases} \]

\( c_0 = 0 \) (average value of \( f(t) \)).

\[ c_n = \frac{1}{2\pi} \int_{-\pi}^\pi f(t) e^{-int} \, dt = \frac{1}{2\pi} \int_{-\pi}^0 -e^{-int} \, dt + \frac{1}{2\pi} \int_0^\pi e^{int} \, dt \]

\[ = \frac{1}{2\pi} \left[ \left. \frac{1}{i} e^{int} \right|_{-\pi}^0 \right] + \frac{1}{2\pi} \left[ \left. \frac{-1}{i} e^{-int} \right|_0^\pi \right] \]

\[ = \frac{1}{2\pi} \left( 1 - e^{i\pi} - e^{-i\pi} + 1 \right) \quad \text{Note: } e^{i\pi} = e^{-i\pi} = (1)^n = (1)^{-n} \]

\[ = \frac{1}{\pi \sin \pi} \left( 1 - (-1)^n \right) \]

Thus, \( f(t) = \sum_{n=1}^{\infty} \frac{1}{\pi \sin \pi} \left( 1 - (-1)^n \right) \left( e^{-int} + e^{int} \right) \)
Exercise 2: Compute the complex Fourier series of the 2π-periodic extension of \( e^t \).

\[
C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^t \, dt = \frac{1}{2\pi} e^\pi \bigg|_{-\pi}^{\pi} = \frac{1}{2\pi} (e^\pi - e^{-\pi})
\]

\[
C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^t \, e^{-int} \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-int)} \, dt = \frac{1}{2\pi(1-int)} e^{(1-int)t} \bigg|_{-\pi}^{\pi}
\]

\[
= \frac{1}{2\pi(1-int)} \left[ e^{(1-int)\pi} - e^{-(1-int)\pi} \right] = \frac{e^{in\pi}}{2\pi(1-int)} \left[ e^\pi - e^{-\pi} \right]
\]

\[
= \frac{(-1)^n}{2\pi(1-int)} \left[ e^\pi - \frac{1}{e^\pi} \right]
\]

**Note:** \( \frac{1}{1-int} = \frac{1}{1-int} + \frac{1+int}{1+int} = \frac{1+int}{1+n^2} \Rightarrow \]

\[
C_n = \frac{(-1)^n (e^\pi - \frac{1}{e^\pi})}{2\pi (1+n^2)} (1+int)
\]

Now, derive the real Fourier series coefficients:

\[
a_n = C_n + C_{-n} = \frac{(-1)^n (e^\pi - \frac{1}{e^\pi})}{\pi (1+n^2)}
\]

\[
b_n = i(C_n - C_{-n}) = \frac{(-1)^n n (e^\pi - \frac{1}{e^\pi})}{\pi (1+n^2)}
\]

**Parseval's identity:** If \( f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \), then

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} (f(t))^2 \, dt = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2.
\]

**Proof:** \[
\frac{1}{\pi} \int_{-\pi}^{\pi} (f(t))^2 \, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left( \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \right) \, dt
\]

\[
= \frac{a_0}{2\pi} \int_{-\pi}^{\pi} f(t) \, dt + \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \, dt
\]

\[
= \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2). \quad \blacksquare
\]
Application: Compute \( \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots \)

let \( f(t) = t \) on \([-\pi, \pi]\). 

\[ a_n = 0 \quad (\text{since } f(t) \text{ is odd}) \]

\[ b_n = \frac{2}{n} (-1)^n \quad (\text{last week}) \]

\( \Rightarrow b_n^2 = \frac{4}{n^2} \)

\[ \frac{1}{\pi} \int_{-\pi}^{\pi} (f(t))^2 \, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \, dt = \frac{2\pi^2}{3} \]

\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty} \frac{4}{n^2} \]

Parsenal \( \Rightarrow \sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{2\pi^2}{3} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \)

Partial differential equations

Let \( u(x,t) \) be a 2-variable function. A \underline{partial differential equation (PDE)} is an equation involving \( u, x, t \), and the partial derivative of \( u \).

Example: \( \frac{du}{dt} = \frac{du^2}{dx^2} \) or just \( u_t = u_{xx} \).

\( \text{ODEs} \) have a unifying theory of existence & uniqueness of solutions.

\( \text{PDEs} \) have no such theory.

\( \text{PDEs} \) arise from physical phenomena & modeling.

Heat equation: \( \rho(x) C(x) \frac{du}{dt} = \frac{d}{dx} \left( x(x) \frac{du}{dx} \right) \), where

\( u(x,t) = \text{temperature of a bar at position } x \text{ & time } t \)
\( \rho(x) = \text{density of bar at position } x \)
\( C(x) = \text{specific heat at position } x \)
\( x(x) = \text{thermal conductivity of the bar} \).
Usually, the bar is homogeneous (i.e., \( p, \sigma, k \) are constants). In this case, the heat equation becomes

\[
\frac{du}{dt} = C^2 \frac{d^2 u}{dx^2}
\]

where \( C = \frac{k}{\rho \sigma} \).

**Example:** Let \( u(x,t) = \text{temp. of a bar of length } \pi, \text{ insulated along the sides, whose ends are kept at zero temp. (Boundary conditions).} \)

and, \( u(x,0) = 3 \sin 2x \) (Initial condition).

Thus, we have the following initial value problem:

\[
U_t = C^2 U_{xx}, \quad U(0,t) = 0, \quad U(\pi,t) = 0, \quad U(x,0) = 3 \sin 2x.
\]

**Note:** This is homogeneous and linear, i.e., if \( u_1, u_2 \) are solutions, then so is \( C_1 u_1 + C_2 u_2 \) (superposition).

**Step 1:** Find the general solution to \( U_t = C^2 U_{xx} \).

* Assume \( u(x,t) = f(x)g(t) \).

\[
U_t = f(x)g'(t), \quad U_{xx} = f''(x)g(t).
\]

Plug back in & solve for \( f \) & \( g \).

\[
U_t = C^2 U_{xx} \quad \Rightarrow \quad f(x)g'(t) = C^2 f''(x)g(t)
\]

\[
\Rightarrow \frac{g'(t)}{C^2 g(t)} = \frac{f''(x)}{f(x)} = \lambda \quad \text{must be constant!}
\]

\( \lambda \) doesn't depend on \( x \), \( \lambda \) doesn't depend on \( t \).
Now, we have two ODEs: \[ \frac{g'(t)}{c^2 g(t)} = \lambda, \quad \frac{f''(x)}{f(x)} = \lambda. \]

Solve for \( g \): \[ g' = c^2 \lambda g \quad \Rightarrow \quad g(t) = A e^{c^2 \lambda t}. \quad \checkmark \]

Suppose \( g(t) \neq 0 \). Boundary condition: \( u(0, t) = u(\pi, t) = 0 \)

becomes \( f(0) g(t) = f(\pi) g(t) = 0 \quad \Rightarrow \quad f(0) = 0 \quad \text{and} \quad f(\pi) = 0. \)

Solve for \( f \): \[ f'' = \lambda f, \quad f(0) = 0, \quad f(\pi) = 0. \]

**Case 1:** \( \lambda = 0 \)

\[ f'' = 0 \quad \Rightarrow \quad f(x) = ax + b. \quad f(0) = 0 \quad \Rightarrow \quad a = 0. \]

\[ f(\pi) = 0 \quad \Rightarrow \quad b = 0 \quad \Rightarrow \quad f(x) = 0. \]

**Case 2:** \( \lambda > 0 \)

\[ f(x) = C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x}. \]

Or \[ f(x) = A \cosh(\sqrt{\lambda} x) + B \sinh(\sqrt{\lambda} x) \quad \text{[This will be easier]} \]

[Recall: \( \cosh 0 = 1 \), \( \sinh 0 = 0 \)]

\[ f(0) = A = 0 \quad \Rightarrow \quad f(x) = B \sinh(\sqrt{\lambda} x) \]

\[ f(\pi) = B \sinh(\sqrt{\lambda} \pi) = 0 \quad \Rightarrow \quad B = 0. \quad \Rightarrow \quad f(x) = 0. \]

**Case 3:** \( \lambda < 0 \) \( \Rightarrow \)

\( w = \sqrt{-\lambda} \), \( f'' = -w^2 f. \)

\[ f(x) = a \cos wx + b \sin wx. \quad f(0) = 0, \quad f(\pi) = 0. \]

\[ f(0) = a = 0 \quad \Rightarrow \quad f(x) = b \sin wx \]

\[ f(\pi) = b \sin \omega \pi = 0 \quad \Rightarrow \quad \omega \pi = n \pi \quad \Rightarrow \quad w = n \quad \text{[\( w = \sqrt{-\lambda} \), so \( \lambda = -n^2 \)]} \]

Recall, \( w = \sqrt{-\lambda} \), so \( \lambda = -n^2 \)

Thus, \( f(x) = B \sin n x \)
Putting this together, for any integer \( n \), we have a sol'n
\[
U_n(x, t) = f_n(x) g_n(t) \quad \text{where} \quad g_n(t) = A_n e^{-c^2 n^2 t} \\
f_n(x) = B_n \sin nx.
\]

Thus, \( U_n(x, t) = A_n e^{-c^2 n^2 t} \sin nx \) is a sol'n for any \( n \).

By superposition, any linear combination is also a solution.

So, the general solution is
\[
U(x, t) = \sum_{n=1}^{\infty} U_n(x, t)
\]
\[
U(x, t) = \sum_{n=1}^{\infty} b_n \sin nx \ e^{-c^2 n^2 t} \quad (\star)
\]

Now, let's solve the initial value problem: \( U(x, 0) = 3 \sin 2x \).

\[
U(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx = 3 \sin 2x \quad \Rightarrow \quad b_2 = 3 \quad \text{and} \quad b_n = 0 \ (n \neq 2).
\]

Thus, \( (\star) \) reduces down to
\[
U(x, t) = \sin 2x \ e^{-4c^2 t}
\]

**Question:** What if instead, we had the initial condition \( U(x, 0) = x(\pi - x) \)?

i.e., we had the following initial value problem:

\[
U_t = c^2 U_{xx}, \quad U(0, t) = u(\pi, t) = 0, \quad U(x, 0) = x(\pi - x).
\]

As before, we get the same general sol'n
\[
U(x, t) = \sum_{n=1}^{\infty} b_n \sin nx \ e^{-c^2 n^2 t}
\]

Plug in \( t = 0 \):
\[
U(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx = x(\pi - x).
\]

To solve for the \( b_n \)'s, we must write \( x(\pi - x) \) as a Fourier sine series.
Recall that in the Fourier sine series of \( x(\pi-x) \),

\[
  b_n = \frac{2}{\pi} \int_0^{\pi} x(\pi-x) \sin nx \, dx = \frac{4}{\pi n^3} (1 - (-1)^n)
\]

Thus, \( u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{4}{\pi n^3} (1 - (-1)^n) \sin nx \)

\( \Rightarrow b_n = \frac{4}{\pi n^3} (1 - (-1)^n) \)

Our particular solution is now

\[
  u(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3} (1 - (-1)^n) \sin nx \, e^{-c^2 n^2 t}
\]

**Note:** In both cases, the steady-state solution is

\[
  \lim_{t \to \infty} u(x,t) = 0.
\]

Algebraically, \( e^{-c^2 n^2 t} \to 0 \)

Physically, heat dissipates.

Now, consider the same problem, but with different boundary conditions:

\[
  U_t = c^2 U_{xx}, \quad U_x(0,t) = U_x(\pi,t) = 0, \quad U(x,0) = X(\pi-x).
\]

This represents insulated endpoints, through which no heat can pass.

\[
  U(x,0) = X(\pi-x)
\]

The steady-state solution is average temperature (as we'll see, is \( \frac{a_0}{2} \) !)

To solve this, proceed as before, but we'll get

\[
  f'' = \lambda f, \quad f'(0) = f'(\pi) = 0 \quad (\text{instead of } f(0) = f(\pi) = 0).
\]

This has solution \( f(x) = a \cos nx + b \sin nx, \quad b = 0 \)

\( \Rightarrow f_n(x) = a_n \cos nx \) for \( n \geq 0 \).
Thus, the general solution becomes

\[ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} f_n(x) g_n(t). \quad \text{(Note: when } n=0, \text{ } f_n \text{ is constant)} \]

\[ \implies u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx e^{-c^2n^2t} \]

To find the particular solution, plug in \( t=0 \):

\[ u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = X(\pi - x) \]

Recall (from HW) : \( X(\pi - x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2} (-1)^n \cos nx \)

Thus, the particular solution is

\[ u(x,t) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2} (-1)^n \cos nx e^{-\frac{c^2t}{t}} \]

Note: The steady-state solution is \( \lim_{t \to \infty} u(x,t) = \frac{\pi^2}{6} \), which is the average temperature.