MTHSC 208, Week 13

Week 12 summary

- For a vector space V, we can define an <u>inner product</u> $\langle\ ,\ \rangle$ (generalized dot product). This allows us to measure vectors (angles & lengths), and project vectors.
- For Per_{2L}, define $\langle f, g\rangle = \frac{1}{L}\int_{-L}^{L} f(t)\, g(t)\, dt$

  * If $f(t)$ is 2π-periodic, then

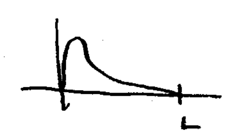
  $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

  $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt\, dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt\, dt.$$

- Even functions:
  - * $f(t) = f(-t)$
  - * Symmetric about y-axis
  - * Fourier series contains only cosine terms
  - * $\int_{-L}^{L} f(t)\, dt = 2\int_0^L f(t)\, dt.$

  Odd functions:
  - * $f(t) = -f(-t)$
  - * Symmetric about origin
  - * Fourier series contains only sine terms
  - * $\int_{-L}^{L} f(t)\, dt = 0.$

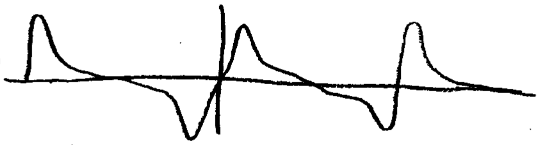
- Fourier cosine & sine series:

  Start with a function $f(t)$ on $[0, L]$, e.g., 

  * <u>Fourier cosine series</u> is the Fourier series of the <u>even extension</u>

  e.g.,  $a_n = \frac{2}{L}\int_0^L f(t) \cos\left(\frac{n\pi t}{L}\right) dt$

* <u>Fourier sine series</u> is the Fourier series of the <u>odd extension</u>

e.g.,  $b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right) dt$.

• Real Fourier series: $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$

Complex Fourier series: $f(t) = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{-int} + c_{-n} e^{int}\right)$

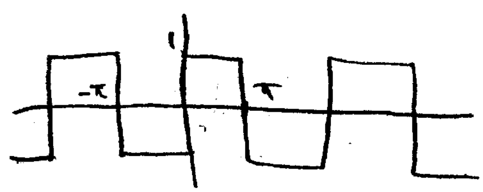
$$= \sum_{n=-\infty}^{\infty} c_n e^{-int}.$$

<u>Recall</u>: $e^{int} = \cos nt + i \sin nt$, $\cos nt = \frac{e^{int} + e^{-int}}{2}$, $\sin nt = \frac{e^{int} - e^{-int}}{2i}$

Thus, $a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$, $c_n = \frac{a_n - ib_n}{2}$, $c_{-n} = \frac{a_n + ib_n}{2}$

* Key idea: $\mathcal{B} = \{e^{-int} : n \in \mathbb{Z}\}$ is a (better) basis for $\mathrm{Per}_{2\pi}$.

<u>Complex Fourier series</u>

<u>Ex 1</u>: Compute the complex Fourier series of 

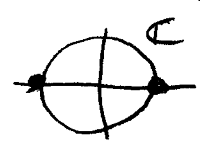
$c_0 = 0$ (average value of $f(t)$).

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt = \frac{1}{2\pi}\int_{-\pi}^{0} -e^{-int}\, dt + \frac{1}{2\pi}\int_0^{\pi} e^{-int}\, dt$$

$$= \frac{1}{2\pi}\left[\frac{1}{in} e^{-int}\Big|_{-\pi}^{0}\right] + \frac{1}{2\pi}\left[\frac{-1}{in} e^{-int}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi i n}\left(1 - e^{in\pi} - e^{-in\pi} + 1\right)$$

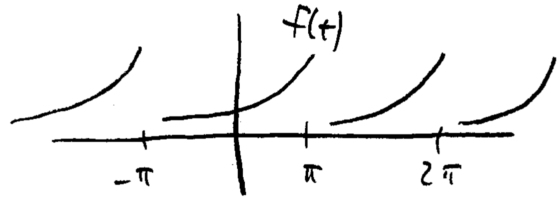
<u>Note</u>: $e^{-in\pi} = e^{in\pi} = (-1)^n = (-1)^{-n}$



$$= \boxed{\frac{1}{\pi i n}\left(1 - (-1)^n\right)}$$

Thus, $\boxed{f(t) = \sum_{n=1}^{\infty} \frac{1}{\pi i n}\left(1 - (-1)^n\right)\left(e^{-int} + e^{int}\right)}$

Ex 2: Compute the complex Fourier series of the 2π-periodic extension of e^t



$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^t\,dt = \frac{1}{2\pi} e^t \Big|_{-\pi}^{\pi} = \boxed{\frac{1}{2\pi}\left(e^{\pi} - e^{-\pi}\right)}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^t e^{-int}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)t}\,dt = \frac{1}{2\pi(1-in)}\, e^{(1-in)t}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(1-in)}\left[e^{(1-in)\pi} - e^{-(1-in)\pi}\right] = \frac{e^{in\pi}}{2\pi(1-in)}\left[e^{\pi} - e^{-\pi}\right]$$

$$= \frac{(-1)^n}{2\pi(1-in)}\left[e^{\pi} - 1/e^{\pi}\right]$$

Note: $\frac{1}{1-in} = \frac{1}{1-in}\,\frac{1+in}{1+in} = \frac{1+in}{1+n^2} \Rightarrow \boxed{c_n = \frac{(-1)^n\left(e^{\pi} - 1/e^{\pi}\right)}{2\pi(1+n^2)}(1+in)}$

Now, derive the real Fourier series coefficients:

$$a_n = c_n + c_{-n} = \frac{(-1)^n(e - 1/e)}{\pi(1+n^2)}$$

$$b_n = i(c_n - c_{-n}) = \frac{(-1)^n\, n\,(e - 1/e)}{\pi(1+n^2)}$$

Parsevals identity: If $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(t))^2\,dt = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Proof: $\frac{1}{\pi}\int_{-\pi}^{\pi}(f(t))^2\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)\right)dt$

$$= \frac{a_0}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)\,dt$$

$$= \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left(a_n \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt + b_n \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt\right).$$

$$= \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2). \qquad \square$$

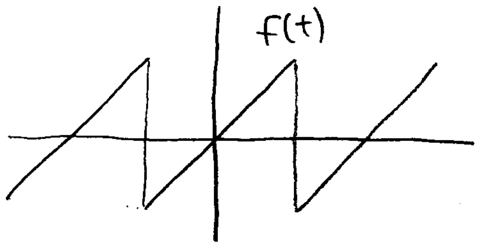
<u>Application</u>: Compute $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$

Let $f(t) = t$ on $[-\pi, \pi]$.

$a_n = 0$ (since $f(t)$ is odd)

$b_n = \frac{2}{n}(-1)^n$ (last week)

$\Rightarrow b_n^2 = \frac{4}{n^2}$.



$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(t))^2\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{2\pi^2}{3}.$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

Parseval $\Rightarrow \sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{2\pi^2}{3} \Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$

Partial differential equations

Let $u(x,t)$ be a 2-variable function. A <u>partial differential equation</u> (PDE) is an equation involving u, x, t, and the partial derivatives of u.

<u>Example</u>: $\frac{\partial u}{\partial t} = \frac{\partial u^2}{\partial x^2}$ or just $u_t = u_{xx}$.

ODE's have a unifying theory of existence & uniqueness of solutions.

PDE's have no such theory.

PDE's arise from physical phenomena & modeling.

<u>Heat equation</u>: $\rho(x)\,\sigma(x)\,\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K(x)\,\frac{\partial u}{\partial x}\right)$, where

$u(x,t)$ = temperature of a bar at position x & time t

$\rho(x)$ = density of bar at position x.

$\sigma(x)$ = specific heat at position x

$K(x)$ = thermal conductivity of the bar.

Usually, the bar is homogeneous (i.e., ρ, σ, κ are constants).

In this case, the heat equation becomes

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}} \qquad \text{where} \quad c = \frac{\kappa}{\rho\sigma}.$$

<u>Example:</u> Let $u(x,t)$ = temp. of a bar of length π, insulated along the sides, whose ends are kept at zero temp. (<u>Boundary</u> conditions).

and, $u(x,0) = 3\sin 2x$ (<u>Initial</u> condition).

Thus, we have the following initial value problem:

$$u_t = c^2 u_{xx} \quad u(0,t) = 0, \quad u(\pi,t) = 0 \quad u(x,0) = 3\sin 2x.$$

<u>Note:</u> This is <u>homogeneous</u> and <u>linear</u>, i.e., if u_1 & u_2 are solutions, then so is $c_1u_1 + c_2u_2$ (superposition).

<u>Step 1:</u> Find the general solution to $u_t = c^2 u_{xx}$.

* Assume $u(x,t) = f(x)\,g(t)$. $\quad u_t = f(x)\,g'(t)$, $\quad u_{xx} = f''(x)\,g(t)$.

Plug back in & solve for f & g.

$$u_t = c^2 u_{xx} \rightsquigarrow f(x)\,g'(t) = c^2 f''(x)\,g(t).$$

$$\Rightarrow \underbrace{\frac{g'(t)}{c^2 g(t)}}_{\text{Doesn't depend on } x} = \underbrace{\frac{f''(x)}{f(x)}}_{\text{Doesn't depend on } t} = \lambda$$

must be <u>constant</u>!

Now, we have 2 ODEs: $\dfrac{g'(t)}{c^2 g(t)} = \lambda$, $\dfrac{f''(x)}{f(x)} = \lambda$.

Solve for g: $g' = c^2\lambda g \Rightarrow g(t) = Ae^{c^2\lambda t}$ ✓

Suppose $g(t) \neq 0$. Boundary condition: $u(0,t) = u(\pi, t) = 0$

becomes $f(0)g(t) = f(\pi)g(t) = 0 \Rightarrow f(0) = 0$ and $f(\pi) = 0$.

Solve for f: $f'' = \lambda f$, $f(0) = 0$, $f(\pi) = 0$.

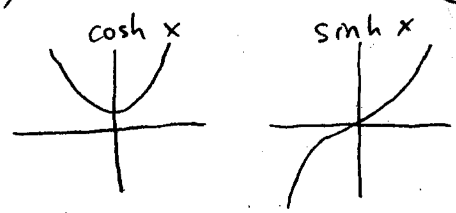
Case 1: $\boxed{\lambda = 0}$. $f'' = 0 \Rightarrow f(x) = ax + b$. $f(0) = 0 \Rightarrow a = 0$.

$f(\pi) = 0 \Rightarrow b = 0 \Rightarrow f(x) = 0$.

Case 2: $\boxed{\lambda > 0}$ $f(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$.

OR $f(x) = A\cosh(\sqrt{\lambda}x) + B\sinh(\sqrt{\lambda}x)$ [This will be easier]

[Recall: $\cosh 0 = 1$, $\sinh 0 = 0$]



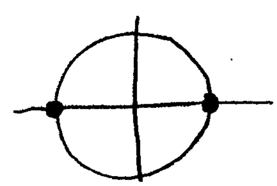
$f(0) = A = 0 \Rightarrow f(x) = B\sinh(\sqrt{\lambda}x)$

$f(\pi) = B\sinh(\sqrt{\lambda}\pi) = 0 \Rightarrow B = 0. \Rightarrow f(x) = 0$.

Case 3: $\boxed{\lambda < 0}$ let $\omega = \sqrt{-\lambda}$, $f'' = -\omega^2 f$.

$f(x) = a\cos\omega x + b\sin\omega x$. $f(0) = 0$, $f(\pi) = 0$.

$f(0) = a = 0 \Rightarrow f(x) = b\sin\omega x$



$f(\pi) = b\sin\omega\pi = 0 \Rightarrow \omega\pi = n\pi$

$\Rightarrow \boxed{\omega = n}$

Recall, $\omega = \sqrt{-\lambda}$, so $\lambda = -n^2$

Thus, $f(x) = B\sin nx$

Putting this together, for <u>any</u> integer n, we have a sol'n

$$u_n(x,t) = f_n(x)\, g_n(t) \quad \text{where} \quad g_n(t) = A_n e^{-c^2 n^2 t}$$

$$f_n(x) = B_n \sin nt$$

Thus, $\boxed{u_n(x,t) = A_n e^{-c^2n^2t} \sin nx}$ is a sol'n for <u>any</u> n.

By superposition, any linear combination is also a solution.

So, the <u>general solution</u> is $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \; e^{-c^2n^2t}} \quad (*)$$

Now, let's solve the initial value problem: $u(x,0) = 3 \sin 2x$.

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx = 3 \sin 2x \Rightarrow b_2 = 3 \text{ and } b_n = 0 \ (n \neq 2).$$

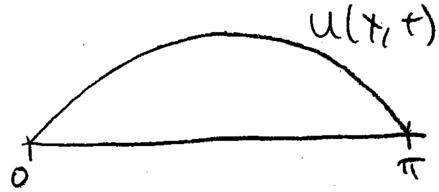
Thus, $(*)$ reduces down to $\boxed{u(x,t) = \sin 2x \; e^{-4c^2t}}$

<u>Question</u>: What if instead, we had the init. condition $u(x,0) = x(\pi - x)$?

i.e, we had the following initial value problem:

$$u_t = c^2 u_{xx}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = x(\pi - x).$$

As before, we get the same gen'l sol'n



$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \; e^{-c^2n^2t}$$

Plug in $t=0$: $u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx = x(\pi - x)$.

To solve for the b_n's, we <u>must</u> write $x(\pi - x)$ as a Fourier sine series

8

Recall that in the Fourier sine series of $x(\pi - x)$,

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi-x)\sin nx \, dx = \frac{4}{\pi n^3}\left(1-(-1)^n\right)$$

Thus, $u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right)\sin nx$

$$\Rightarrow b_n = \frac{4}{\pi n^3}\left(1-(-1)^n\right)$$

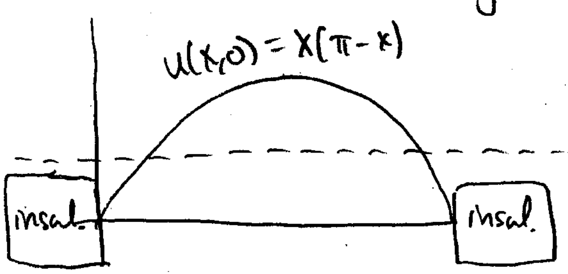
Our particular sol'n is now $\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right)\sin nx \, e^{-c^2n^2t}}$

Note: In both cases, the steady-state solution is $\lim_{t\to\infty} u(x,t) = 0$. Algebraically, $e^{-c^2n^2t} \longrightarrow 0$.

Physically, heat dissipates.

Now, consider the same problem, but with different boundary conds:

$$u_t = c^2 u_{xx}, \quad \underbrace{u_x(0,t) = u_x(\pi,t) = 0}_{\substack{\text{represents insulated endpoints,} \\ \text{through which no heat can pass.}}}, \quad u(x,0) = x(\pi - x).$$



steady-state sol'n = average temperature (as we'll see, is $\frac{a_0}{2}$!)

To solve this, proceed as before, but we'll get

$f'' = \lambda f$, $f'(0) = f'(\pi) = 0$ (instead of $f(0) = f(\pi) = 0$).

This has solution $f(x) = a\cos nx + b\sin nx$, $b = 0$

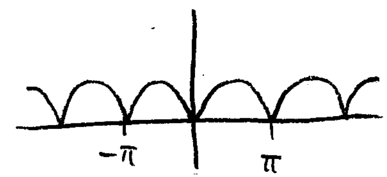
$$\Rightarrow \boxed{f_n(x) = a_n \cos nx} \quad \text{for } n \geq 0.$$

Thus, the general solution becomes

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} f_n(x)\, g_n(t).$$ (Note: when $n=0$, f_n & g_n are constants)

$$\Rightarrow \boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \, e^{-c^2 n^2 t}}$$

To find the particular sol'n, plug in $t=0$:



$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \underbrace{x(\pi - x)}_{\text{Must express as a Fourier cosine series}}.$$

Recall (from HW) : $x(\pi - x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}\left(1-(-1)^n\right) \cos nx$

Thus, the particular sol'n is $\boxed{u(x,t) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}\left(1-(-1)^n\right) \cos nx \, e^{-c^2 n^2 t}}$

Note: The steady-state sol'n is $\lim_{t\to\infty} u(x,t) = \frac{\pi^2}{6}$, which is the average temperature.