MTHSC 208, Week 14

Week 13 summary

- Partial differential equations (PDEs): equations involving a function and its partial derivatives.

* <u>Heat equation:</u> $u_t = c^2 u_{xx}$

Boundary conditions:
- Dirichlet: $u(0,t) = u(L,t) = 0$ (temp. of endpoints fixed at 0)
- Neumann: $u_x(0,t) = u_x(L,t) = 0$ (insulated endpoints).

Initial conditions: $u(x,0) = h(x)$: (Initial heat distribution of bar)

* <u>Solving PDEs by separation of variables:</u>

<u>Step 1</u>: Assume $u(x,t) = f(x)\,g(t)$. Plug back in & separate variables.

<u>Step 2</u>: Set resulting eq'n to constant λ. Get 2 ODEs, one for f & one for g.

<u>Step 3</u>: Solve ODE for g (general sol'n). Solve ODE for f with boundary conditions (particular sol'n). Find λ.

<u>Step 4</u>: General solution is $u(x,t) = \sum_{n=0}^{\infty} f_n(x)\, g_n(t)$ (Superposition).

<u>Step 5</u>: Plug in $t=0$ & use initial conditions to find particular sol'n ($u(x,0)$ must be a Fourier series).

Wave equation:

Consider the following PDE: $\boxed{\dfrac{\partial u}{\partial t} - c\dfrac{\partial u}{\partial x} = 0}$ $(*)$

Let f be any one-variable function, and set

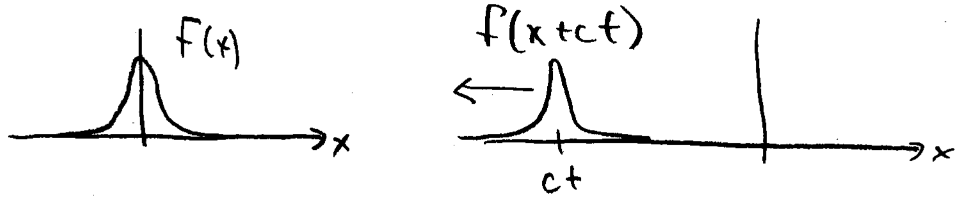
$u(x,t) = f(x+ct)$. $\qquad u_x(x,t) = f'(x+ct)$

$\qquad\qquad\qquad\qquad u_t(x,t) = cf'(x+ct)$ (chain rule!)

Note: $u_t - cu_x = cf'(x+ct) - cf'(x+ct) = 0$ ✓

i.e., $f(x+ct)$ is a solution to $(*)$

Picture of this:



As t increases, $u(x,t) = f(x+ct)$ is a traveling wave (to the left).

Next, consider the PDE: $\boxed{\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0}$ $(**)$

Let g be any one-variable function, and set

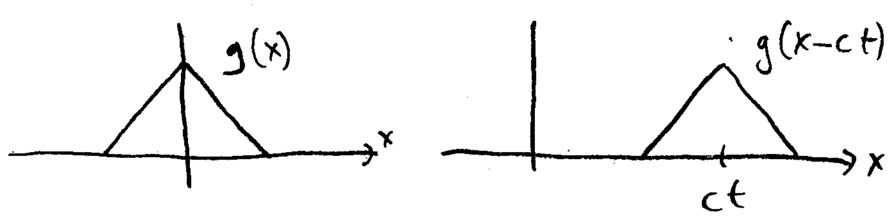
$u(x,t) = g(x-ct)$. $\qquad u_x(x,t) = f'(x-ct)$

$\qquad\qquad\qquad\qquad u_t(x,t) = -cf'(x-ct)$.

Note: $u_t + cu_x = -cg'(x-ct) + cg(x-ct) = 0$. ✓

i.e., $g(x-ct)$ is a solution to $(**)$

Picture of this:



Now, let f & g be any two functions. Consider the PDE

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t}-c\frac{\partial u}{\partial x}\right)=\boxed{\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0} \quad (***)$$

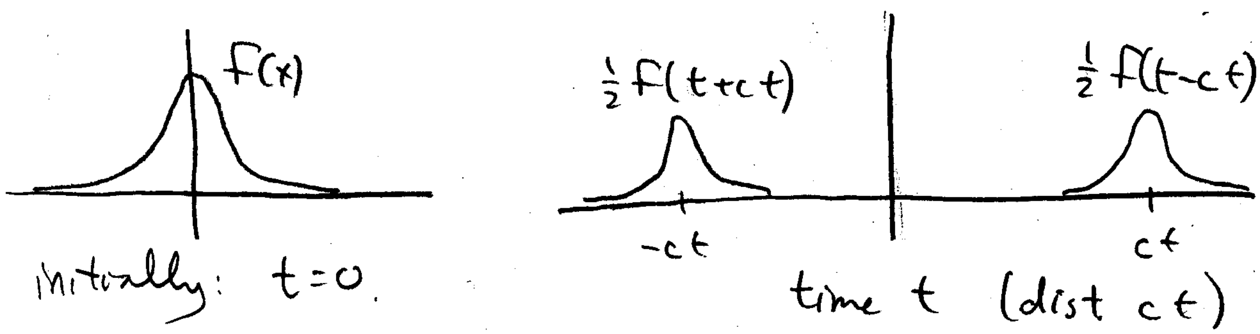
Check: $u(x,t)=f(x+ct)+g(x-ct)$ is a solution.

Consider the following initial conditions with $(***)$:

$u(x,0)=f(x)$ "initial displacement"

$u_t(x,0)=0$ "initial velocity, i.e., driving force $=0$."

Think: Start with a wave in the ocean at time $t=0$:



The solution to this initial value problem is

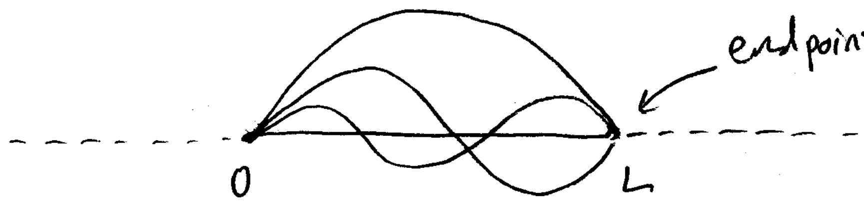
$$u(x,t)=\tfrac{1}{2}f(x+ct)+\tfrac{1}{2}f(x-ct)$$

"half the wave (or energy) goes to the left, half goes right."

Moral: $\boxed{\dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2} \text{ is the } \underline{\text{wave equation}}}$

- Now, suppose we want to model vibrations (waves) on a finite string/wire of length L.

We need to impose boundary conditions.



endpoints are fixed for all t.

Let $u(x,t)$ be the (vertical) displacement at point x & time t.

Fixed endpoints $\Rightarrow$ $u(0,t)=0$, $u(L,t)=0$.

Must specify initial wave: $u(x,0)=h_1(x)$

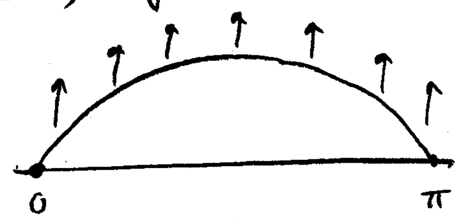
and initial velocity @ x: $u_t(x,0)=h_2(x)$.

(up/down, not left/right).

Together, we get an initial value problem for the wave equation:

$$\boxed{u_{tt}=c^2u_{xx} \qquad u(0,t)=0, \quad u(L,t)=0 \qquad u(x,0)=h_1(x), \quad u_t(x,0)=h_2(x)}$$

We solve this using separation of variables, just like the heat equation.



Example:

$$u_{tt}=c^2u_{xx}$$

$$u(0,t)=0, \quad u(\pi,t)=0$$

$$u(x,0)=x(\pi-x), \quad u_t(x,0)=1$$

Assume $u(x,t)=f(x)\,g(t)$. Plug back in:

$$u_{tt}=fg'', \quad u_{xx}=f''g \Rightarrow fg''=c^2f''g$$

Thus, $\frac{f''}{f} = \frac{g''}{c^2 g} = \lambda \Rightarrow \begin{cases} f'' = \lambda f \\ g'' = c^2 \lambda g \end{cases}$

Moreover, $u(0,t) = f(0)\,g(t) = 0 \Rightarrow f(0) = 0$

$u(\pi,t) = f(\pi)\,g(t) = 0 \Rightarrow f(\pi) = 0$

<u>ODE 1:</u> $f'' = \lambda f, \quad f(0) = 0, \quad f(\pi) = 0$

We've done this! $\lambda = -n^2$, $\boxed{f_n(x) = b_n \sin nx}$

<u>ODE 2:</u> $g'' = -c^2 n^2 g \Rightarrow \boxed{g_n(t) = a_n \cos(cnt) + b_n \sin(cnt)}$

Note: for each n, we have a solution

$$u_n(x,t) = f_n(x)\,g_n(t) = (a_n \cos cnt + b_n \sin cnt)\sin nx$$

By superposition, the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x)\,g_n(t) = \boxed{\sum_{n=1}^{\infty} \left(a_n \cos(cnt) + b_n \sin(cnt)\right)\sin nx}$$

Finally, let's use our initial conditions:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx = x(\pi - x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right)\sin nx$$

$\Rightarrow a_n = \frac{4}{\pi n^3}\left(1-(-1)^n\right)$ (Fourier sine series of $x(\pi - x)$).

$$u_t(x,t) = \sum_{n=1}^{\infty}\left(-cn\,a_n \sin(cnt) + cn\,b_n \cos(cnt)\right)\sin nx$$

$$u_t(x,0) = \sum_{n=1}^{\infty} cn\,b_n \sin nx = 1 = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1-(-1)^n\right)\sin nx$$

(Fourier sine series of 1).

$\Rightarrow cn\,b_n = \frac{2}{n\pi}\left(1-(-1)^n\right) \Rightarrow b_n = \frac{2}{cn^2\pi}\left(1-(-1)^n\right)$.

Now, our particular solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[\frac{4}{\pi n^3}\left(1-(-1)^n\right) \cos(cnt) + \frac{2}{\pi c n^2}\left(1-(-1)^n\right) \sin(cnt) \right] \sin nx}$$

PDEs in higher dimensions

Recall PDEs in 1 (spatial) dimension:

* Heat equation: $u_t = c^2 u_{xx}$
* Wave equation: $u_{tt} = c^2 u_{xx}$

In 2 dimensions, these PDEs are

* Heat equation: $u_t = c^2(u_{xx} + u_{yy})$
* Wave equation: $u_{tt} = c^2(u_{xx} + u_{yy})$.

Let $u(x_1, \dots, x_n, t)$ be a function in n (spatial) variables. The Laplacian of u is $\nabla \cdot \nabla u = \boxed{\nabla^2 u = \sum_{n=1}^{n} \frac{\partial^2 u}{\partial x_i}}$

(Recall that $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$, so $\nabla \cdot \nabla = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$).

In n dimensions, these PDEs are

* Heat equation: $u_t = c^2 \nabla^2 u$
* Wave equation: $u_{tt} = c^2 \nabla^2 u$

(Note: Sometimes, the Laplace operator ∇^2 is written Δ).

<u>Steady-state solutions</u>: Occur for the heat equation, but not for the wave equation (heat diffuses, waves propogate).

Note that "steady-state" means that $u_t = 0$, Sol'ns to the heat equation approach this steady-state sol'n, because "eventually, the temperature doesn't change w.r.t. time."

Thus, all steady-state solutions satisfy $0 = u_t = c^2 \nabla^2 u$,

i.e, $\nabla^2 u = 0 \Rightarrow \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \cdots + \dfrac{\partial^2 u}{\partial x_n^2} = 0$.

<u>Def</u>: A function is <u>harmonic</u> if $\nabla^2 u = 0$.

<u>Ex</u>: $f(x,y) = x^2 - y^2$. $f_{xx} = 2$, $f_{yy} = -2$, $\nabla^2 f = 2 - 2 = 0$.

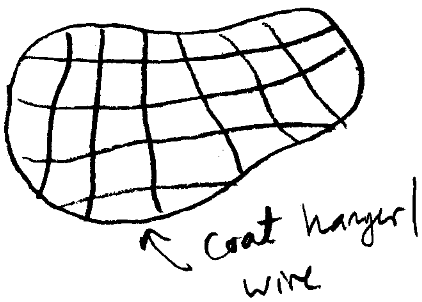
<u>Visualizing harmonic functions</u>:

If $u(x,t)$ is a solution to the heat equation, then $\lim_{t \to \infty} \frac{\partial u}{\partial t} = 0$. "Temperature will spread out evenly."

* Thus, steady-state solutions to the heat equation are harmonic functions, and are as "flat as possible."

<u>In 1D</u>: Consider the temperature $u(x,t)$ of a bar, with $u(0,t) = 0$, $u(L,t) = 100$. The steady-state sol'n satisfies $0 = u_t = c^2 u_{xx}$, so it is a straight line, regardless of init cond.

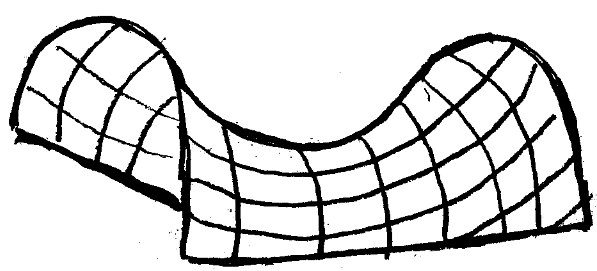
<u>Physical interpretation</u>: Stretch out plastic wrap over a bent circular wire, as tight as possible.



∗ The surface is a <u>harmonic function</u>!

<u>Fact</u>: If f is harmonic, then for any closed bounded region R, f achieves its min & max values on the <u>boundary</u>, ∂R.

<u>Ex</u>: $f(x) = x^2 - y^2$



Picture cutting this surface with a "cookie cutter". The max & min points will be on the boundary.

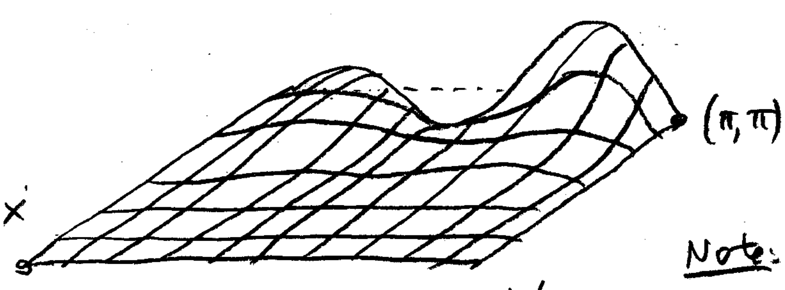
∗ In other words, there are no local min/max's.

<u>Example</u>: Let $u(x,y)$ be the steady-state temperature, $0 \le x, y \le \pi$,

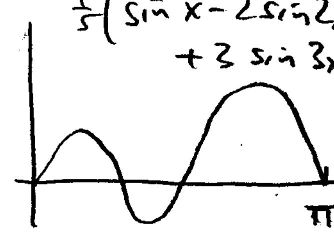
$$u_{xx} + u_{yy} = 0$$

$$u(0,y) = u(\pi,y) = u(x,0) = 0$$

$$u(x,\pi) = \tfrac{1}{5}\sin x - \tfrac{2}{5}\sin 2x + \tfrac{3}{5}\sin 3x$$



<u>Note</u>: $\frac{1}{5}(\sin x - 2\sin 2x + 3\sin 3x)$.



<u>Physical situation</u>: $u(x,y)$ is the steady-state sol'n for region $[0,\pi]\times[0,\pi]$ where 3 sides are fixed at 0°, and one at $u(x,\pi)$.