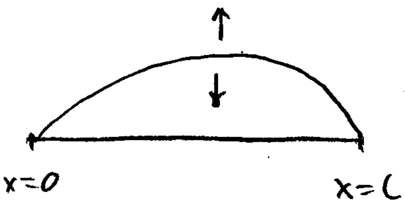
MTHSC 208, Week 15

Week 14 summary



- Wave equation: $u_{tt} = c^2 u_{xx}$

  Boundary conditions: $u(0,t) = u(L,t) = 0$

  Initial conditions: $u(x,0) = h_1(x)$

  $u_t(x,0) = h_2(x)$.

  General sol'n: $u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos cnt + b_n \sin cnt\right) \sin nx.$

- Higher-dimensional PDE's:

  * Heat equation: $u_t = c^2(\nabla^2 u)$ (In 2D: $u_t = c^2(u_{xx} + u_{yy})$)

  * Wave equation: $u_{tt} = c^2(\nabla^2 u)$ (In 2D: $u_{tt} = c^2(u_{xx} + u_{yy})$).

  where $\nabla^2 u(x_1, \ldots, x_n, t) = \dfrac{\partial^2 u}{\partial x_1^2} + \cdots + \dfrac{\partial^2 u}{\partial x_n^2}$

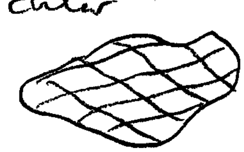
- u is harmonic if $\nabla^2 u = 0$ (this is Laplace's equation)

  Harmonic functions:

  * Have no local mins/maxes (are "as flat as possible")

  * Are precisely the steady-state sol'ns to the heat equation (because $u_t = 0$ in the steady-state)

  * "Look like plastic wrap stretched over a circular bent wire (in 2-D)."

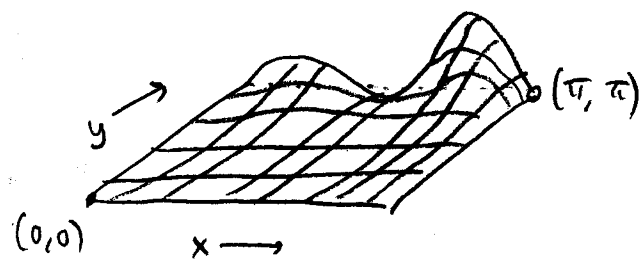


Example : Solving Laplace's equation (cont. from last week).

Consider the PDE: $u_{xx} + u_{yy} = 0$

$u(0,y) = u(\pi,y) = u(x,0) = 0$

$u(x,\pi) = \frac{1}{5}\sin x - \frac{2}{5}\sin 2x + \frac{3}{5}\sin 3x$



Step 1: Assume $u(x,t) = X(x)\,Y(y)$. Plug back in:

$u_{xx} = X''Y, \quad u_{yy} = XY''$

$u_{xx} + u_{yy} = X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = \frac{-Y''}{Y} = \lambda$

Also, $u(0,y) = X(0)\,Y(y) \Rightarrow X(0) = 0$

$u(\pi,y) = X(\pi)\,Y(y) \Rightarrow X(\pi) = 0$

$u(x,0) = X(x)\,Y(0) \Rightarrow Y(0) = 0$

Get 2 ODE's: (i) $X'' = \lambda X, \quad X(0) = X(\pi) = 0$

(ii) $Y'' = -\lambda Y, \quad Y(0) = 0.$

Step 2. Solve for X and Y.

(i) We've done this before. $\lambda = -n^2$, $\boxed{X_n(x) = b_n \sin nx}$

(ii) $Y'' = n^2 Y, \quad Y(0) = 0.$

$Y(y) = A_n \cosh ny + B_n \sinh ny$

$Y(0) = A_n = 0 \quad \Rightarrow \quad \boxed{Y(y) = B_n \sinh ny}$

Step 3: The general sol'n is thus

$u(x,t) = \sum_{n=0}^{\infty} X_n(x)\,Y_n(y) = \boxed{\sum_{n=1}^{\infty} b_n \sin nx \sinh ny}$

Step 4: Use the last boundary condition.

$$u(x,\pi) = \sum_{n=1}^{\infty} (b_n \sinh n\pi) \sin nx = \frac{1}{5}\sin x - \frac{2}{5}\sin 2x + \frac{3}{5}\sin 3x$$

$n=1$: $b_1 \sinh \pi = \frac{1}{5} \Rightarrow b_1 = \frac{1}{5\sinh \pi}$

$n=2$: $b_2 \sinh 2\pi = -\frac{2}{5} \Rightarrow b_2 = \frac{-2}{5\sinh 2\pi}$

$n=3$: $b_3 \sinh 3\pi = \frac{3}{5} \Rightarrow b_3 = \frac{3}{5\sinh 3\pi}$

Our particular sol'n is therefore

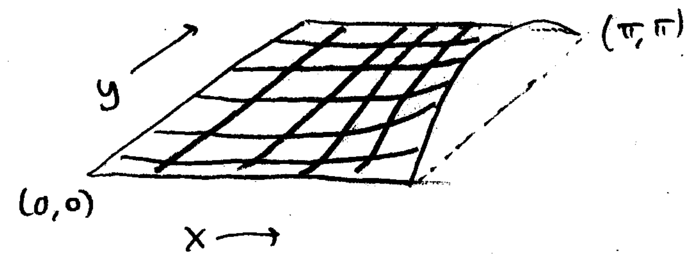
$$\boxed{u(x,y) = \frac{1}{5\sinh\pi}\sin x \sinh y - \frac{2}{5\sinh 2\pi}\sin 2x \sinh 2y + \frac{3}{5\sinh 3\pi}\sin 3x \sinh 3y}$$

Example 2: Consider a similar PDE:

$u_{xx} + u_{yy} = 0$

$u(0,y) = 0, \quad u(\pi,y) = y(\pi - y)$

$u(x,0) = u(x,\pi) = 0$



Proceed as before, and get

$Y'' = \lambda Y, \quad Y(0) = Y(\pi) = 0 \Rightarrow Y_n(y) = b_n \sin ny$

$X'' = n^2 X, \quad X(0) = 0 \qquad \Rightarrow X_n(x) = B_n \sinh nx$

Gen'l sol'n: $u(x,y) = \sum_{n=1}^{\infty} b_n \sinh nx \sin ny$

Init. cond: $u(\pi,y) = y(\pi-y) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}(1-(-1)^n)\sin ny = \sum_{n=1}^{\infty}(b_n \sinh n\pi)\sin ny$

$\Rightarrow b_n \sinh n\pi = \frac{4}{\pi n^3}(1-(-1)^n) \Rightarrow b_n = \frac{4(1-(-1)^n)}{\pi n^3 \sinh n\pi}$

Particular sol'n: $\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3 \sinh n\pi} \sinh nx \sin ny}$

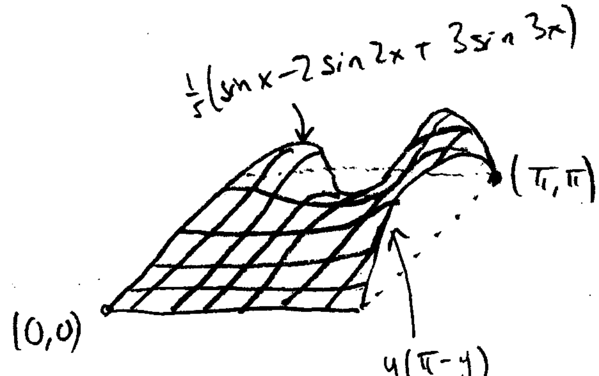
<u>Example 3</u>: Consider a superposition of the 2 previous examples.

$$u_{xx} + u_{yy} = 0$$

$$u(0,y) = 0, \quad u(x,0) = 0$$

$$u(\pi, y) = \frac{1}{5}\sin x - \frac{2}{5}\sin 2x + \frac{3}{5}\sin 3x$$

$$u(x,\pi) = y(\pi - y).$$



Intuitively, the solution $u(x,y)$ (think: steady-state sol'n of these 2 PDE's; superimposed boundary conditions) should be the superposition (<u>sum</u>) of the 2 solutions from Examples 1 & 2

i.e.,

Thus, the particular sol'n is

$$\boxed{\begin{aligned} u(x,y) = &\left(\frac{1}{5\sinh\pi}\sin x \sinh y + \frac{8}{\pi\sinh\pi}\sinh x \sin y\right) \\ &- \left(\frac{2}{5\sinh 2\pi}\sin 2x \sinh 2y\right) \\ &+ \left(\frac{3}{5\sinh 3\pi}\sin 3x \sinh 3y + \frac{8}{27\pi\sinh 3\pi}\sinh 3x \sin 3y\right) \\ &+ \sum_{n=4}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3 \sinh n\pi}\sin nx \sinh ny. \end{aligned}}$$

<u>Note</u>: This can all be done more generally in a rectangular $L_1 \times L_2$ region, instead of just $\pi \times \pi$.

<u>Heat equation in 2D</u>: (Example) : $u_t = c^2(u_{xx}+u_{yy})$

Let $u(x,y,t)$ = temp of a square region, $0 \le x, y \le \pi$.

$u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0$ (<u>Boundary</u> fixed at $0°$).

$u(x,y,0) = 2\sin x \sin 2y + 3\sin 4x \sin 5y$ (<u>Init.</u> heat distribution).

Assume sol'n has the form $u(x,y,t) = \underbrace{f(x,y)}_{\text{f}^n \text{ of pos.}}\,\underbrace{g(t)}_{\text{f}^n \text{ of time.}}$

Plug back in:

$$u_t = c^2(u_{xx}+u_{yy}) \Rightarrow g'f = c^2 g f_{xx} + c^2 g f_{yy}$$

$$\Rightarrow \frac{g'}{c^2 g} = \frac{f_{xx}+f_{yy}}{f} = \frac{\nabla^2 f}{f} = \lambda.$$

Get 2 equations: $\boxed{\begin{array}{l} g' = c^2\lambda g \\ \nabla^2 f = \lambda f \end{array}}$ $\Rightarrow g(t) = C e^{c^2\lambda t}$

"Helmholz equation."

Boundary conditions on f: $u(0,y,t) = f(0,y)\,g(t) = 0 \Rightarrow f(0,y) = 0$

$u(\pi,y,t) = f(\pi,y)\,g(t) = 0 \Rightarrow f(\pi,y) = 0.$

Need to solve: $f_{xx}+f_{yy} = \lambda f$, $f(0,y) = f(\pi,y) = 0$

Assume $f(x,y) = X(x)\,Y(y)$. $f_{xx} = X''Y$, $f_{yy} = XY''$

Plug back in: $\dfrac{X''Y + XY''}{XY} = \dfrac{\lambda XY}{XY} \Rightarrow \dfrac{X''}{X} + \dfrac{Y''}{Y} = \lambda$

$$\Rightarrow \underbrace{\frac{X''}{X}}_{\text{depends only on } x} = \underbrace{\lambda - \frac{Y''}{Y}}_{\text{depends only on } y} = \mu \quad \leftarrow \text{const.}$$

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Get 2 ODE's:

$$X'' = \mu X, \qquad Y'' = (\overbrace{\lambda - \mu}^{\text{call this } \nu}) Y \quad \Rightarrow \quad \lambda - \mu = \nu \Rightarrow \lambda = \nu + \mu.$$

$$Y'' = \nu$$

Recall boundary conditions:

$$f(0, y) = X(0)Y(y) = 0 \Rightarrow X(0) = 0$$
$$f(\pi, y) = X(\pi)Y(y) = 0 \Rightarrow X(\pi) = 0$$
$$f(x, 0) = X(x)Y(0) = 0 \Rightarrow Y(0) = 0$$
$$f(x, \pi) = X(x)Y(\pi) = 0 \Rightarrow Y(\pi) = 0$$

ODE 1: $X'' = \mu X, \quad X(0) = X(\pi) = 0 \Rightarrow X_n(x) = b_n \sin nx, \quad \mu = -n^2$

ODE 2: $Y'' = \nu Y, \quad Y(0) = Y(\pi) = 0 \Rightarrow Y_m(y) = B_m \sin mx, \quad \nu = -m^2$

Recall: $\lambda = \nu + \mu = -(n^2 + m^2).$

Thus, for each pair m & n, we have a sol'n

$$\boxed{f_{nm}(x, y) = b_{nm} \sin nx \sin my}$$

Also, $g(t) = Ce^{\lambda t} \Rightarrow \boxed{g_{nm}(t) = C_{nm} e^{-c^2(n^2+m^2)t}}$

Thus, the general sol'n is

$$\boxed{u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{nm} \sin nx \sin my \; e^{-c^2(n^2+m^2)t}}$$

Use init. cond: $u(x, y, 0) = \sum_{n,m=1}^{\infty} b_{nm} \sin nx \sin my$

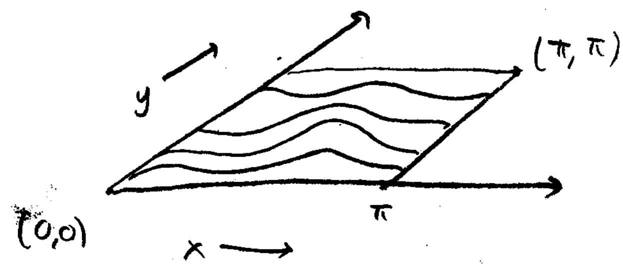
$$= 2 \sin x \sin 2y + 3 \sin 4x \sin 5y$$

$\Rightarrow b_{12} = 2, \quad b_{45} = 3, \quad$ all other $b_{nm} = 0.$

Particular sol'n: $\boxed{u(x, y, t) = 2 \sin x \sin 2y \; e^{-5c^2 t} + 3 \sin 4x \sin 5y \; e^{-41c^2 t}}$

Wave equation in 2D (Example): $u_{tt} = c^2(u_{xx} + u_{yy})$

Let $u(x,y,t)$ = displacement of a square membrane of side length π.



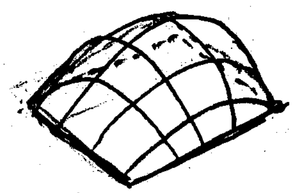
$u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0$ (Boundary is immobile)

$u(x,y,0) = p(x)\,q(y)$ Initial wave (displacement)

$u_t(x,y,0) = 0$ Initial wave velocity (vertically)

Let's solve this if $p(x) = x(\pi - x)$

$q(y) = y(\pi - y)$



Initial wave: "paraboloid-like"

* Solving this is almost the same as solving the 2D heat eq'n. The only difference is $g_{nm}(t)$!

Assume $u(x,y,t) = f(x,y)\,g(t)$.

Plug back in: $fg'' = c^2 g f_{xx} + c^2 g f_{yy}$

$$\Rightarrow \frac{g''}{c^2 g} = \frac{f_{xx} + f_{yy}}{f} = \frac{\nabla^2 f}{f} = \lambda$$

Get 2 equations: $\boxed{\begin{aligned} g'' &= c^2 \lambda g \\ \nabla^2 f &= \lambda f \end{aligned}}$ ← Was $g' = c^2 \lambda g$ in heat eq'n. ← Same as in heat eq'n!

Init cond on g: $u_t(x,y,0) = f(x,y)\,g'(0) = 0 \Rightarrow \boxed{g'(0) = 0.}$

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Solve for f: $\nabla^2 f = \lambda f$, $f(0,y) = f(\pi,y) = f(x,0) = f(x,\pi) = 0$

$\lambda = -(n^2+m^2)$, $\boxed{f_{nm}(x,y) = b_{nm} \sin nx \sin mx}$

(Same as in 2D Heat equation).

Solve for g: $g'' = c^2 \lambda g$ $\quad g'(0) = 0$

$g'' = -c^2(n^2+m^2) g$

$\Rightarrow g_{nm}(t) = A_{nm} \cos(c\sqrt{n^2+m^2}\, t) + B_{nm} \sin(c\sqrt{n^2+m^2}\, t)$

$g'(0) = 0 \Rightarrow B_{nm} = 0$

$\Rightarrow \boxed{g_{nm}(t) = A_{nm} \cos(c\sqrt{n^2+m^2}\, t)}$

For each choice of n & m, we have a sol'n to the wave equation

$u_{nm}(x,y,t) = f_{nm}(x,y)\, g_{nm}(t) = b_{nm} \sin nx \sin mx \cos(c\sqrt{n^2+m^2}\, t)$.

Thus, the general sol'n is

$$\boxed{u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{nm} \sin nx \sin mx \cos(c\sqrt{n^2+m^2}\, t)}$$

Note: Alternatively, we could just write $\sum_{n,m \geq 1} u_{nm}(x,y,t)$.

Use init cond:

Special topics

(1) <u>Laplace transforms:</u> $\mathcal{L}\{y(t)\}(s) = \int_0^\infty y(t)\, e^{-st}\, dt = Y(s)$.

"Turns time-derivatives into multiplication by s:

$$\mathcal{L}\{y'(t)\}(s) = s\,Y(s) - y(0)$$

$$\mathcal{L}\{y''(t)\}(s) = s^2 Y(s) - s\,y(0) - y'(0), \quad \text{and so on.}$$

We can also take $\mathcal{L}\{u(x,t)\}(x,s) = \int_0^\infty u(x,t)\, e^{-st} dt = U(x,s)$.

similarly, $\mathcal{L}\{u_t(x,t)\}(x,s) = s\,U(x,0) - u(x,0)$

$$\mathcal{L}\{u_{tt}(x,t)\}(x,s) = s^2 U(x,s) - s\,u(x,0) - u_t(x,0)$$

$$\mathcal{L}\{u_x(x,t)\}(x,s) = U_x(x,s).$$

$*$ $\mathcal{L}$ turns: ODEs into algebraic equations.

PDEs into ODE (if e.g., $u(x,t)$).

$$\begin{tikzcd}
\text{PDE} \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & \text{ODE} \arrow[d, "\text{solve}"] \\
\text{PDE solution} & \text{ODE solution} \arrow[l, "\mathcal{L}^{-1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{ODE} \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & \text{algebraic equation} \arrow[d] \\
\text{ODE solution} & \text{solution} \arrow[l, "\mathcal{L}^{-1}"]
\end{tikzcd}$$

(2) <u>Fourier transforms:</u> $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x} dx$ (or $\mathcal{F}(\xi)$).

"Turns spacial-derivatives into multiplication by $2\pi i \xi$.

$$\widehat{f'(x)} = 2\pi i \xi\, \hat{f}(\xi), \quad \text{or more generally,}$$

$$\widehat{f^{(n)}(x)} = (2\pi i \xi)^n \hat{f}(\xi)$$

Similarly, $\widehat{u_x(x,t)} = (2\pi i \xi)\, U(\xi, t)$, etc

$\widehat{u_t(x,t)} = U_t(\xi, t)$.

* Fourier transforms turn PDEs into ODEs.

$$\begin{array}{ccc} \text{PDE} & \xrightarrow{\ \mathcal{F}\ } & \text{ODE} \\ \Big\downarrow & & \Big\downarrow \text{solve} \\ \text{PDE solution} & \xleftarrow{\ \mathcal{F}^{-1}\ } & \text{ODE solution} \end{array}$$

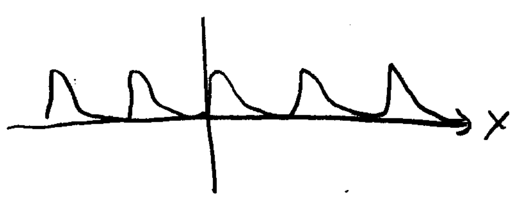
And, the inverse Fourier transform is easy! It's (almost) the same.

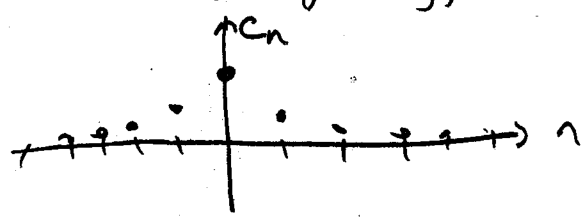
$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi.$$

A neat way to think of Fourier (& Laplace) transforms.

Let $f(x)$ be 2π-periodic. We can plot f as a function...

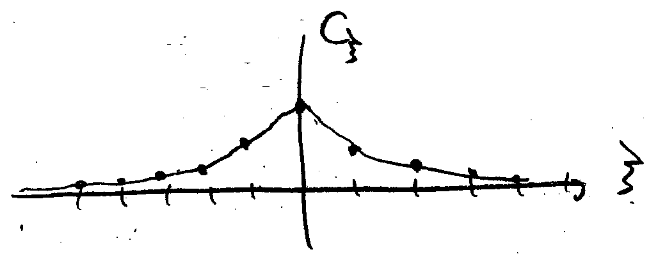
of x (space) <u>OR</u> of n (frequency).





<u>Question</u>: What would a function look like if we allowed a continuum of frequencies

It need not even be periodic.



$$\sum_{n=-\infty}^{\infty} c_n e^{-inx} \quad \text{(function of } n \text{ "frequency")}$$

Start with $f(x)$

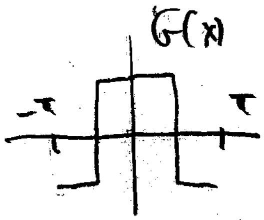
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, d\xi$$

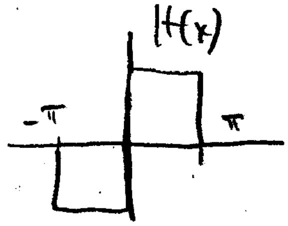
(function of ξ) "frequency"

(3) Wavelets:

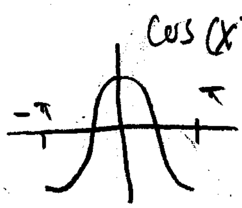
*Big idea: Many periodic functions aren't smooth, so representing them as sines & cosines isn't natural, or efficient (for approximation purposes).

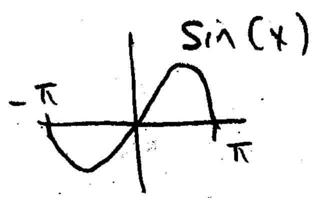
Solution: use a different basis for $Per_{2\pi}$.

e.g., 

& 

instead of

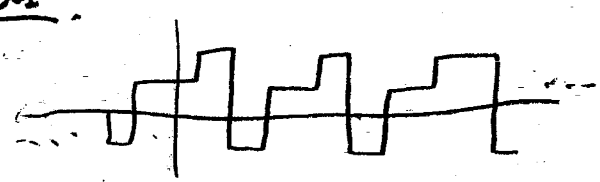




Our new basis is $\{G(nx),\ H(mx),\ n \geq 0,\ m \geq 1\}$

instead of $\{\cos(nx),\ \sin(nx),\ n \geq 0,\ m \geq 1\}$.

These are called Haar wavelets.

They're better to express, e.g., 

Wavelets are used in the new & improved JPEG 2000 file format.