**Week 14 Summary**

- **Wave equation**: \( U_{tt} = c^2 U_{xx} \)
  - **Boundary conditions**: \( U(0, t) = U(L, t) = 0 \)
  - **Initial conditions**: \( U(x, 0) = h_1(x) \)
    \( U_t(x, 0) = h_2(x) \)
  - **General solution**: \( U(x, t) = \sum_{n=1}^{\infty} \left( a_n \cos n\pi t + b_n \sin n\pi t \right) \sin n\pi x \)

- **Higher-dimensional PDEs**:
  - **Heat equation**: \( U_t = c^2 (\nabla^2 U) \) (in 2D: \( U_{tt} = c^2 (U_{xx} + U_{yy}) \))
  - **Wave equation**: \( U_{tt} = c^2 (\nabla^2 U) \) (in 2D: \( U_{tt} = c^2 (U_{xx} + U_{yy}) \))
  - Where \( \nabla^2 U(x_1, \ldots, x_n, t) = \frac{\partial^2 U}{\partial x_1^2} + \ldots + \frac{\partial^2 U}{\partial x_n^2} \)

- **U is harmonic if** \( \nabla^2 U = 0 \) (this is Laplace's equation)

**Harmonic Functions**:
- Have no local mins/amps (are "as flat as possible")
- Are precisely the steady-state solutions to the heat equation (because \( U_t = 0 \) in the steady-state)
- "Look like plastic wrap stretched over a circular bent wire (in 2-D)."
Example: Solving Laplace's equation (cont. from last week).

Consider the PDE: \( U_{xx} + U_{yy} = 0 \)

- \( U(0, y) = U(\pi, y) = U(x, 0) = 0 \)
- \( U(x, \pi) = \frac{1}{3} \sin x - \frac{2}{5} \sin 2x + \frac{2}{5} \sin 3x \)

**Step 1.** Assume \( U(x, y) = X(x) Y(y) \). Plug back in:

\[ U_{xx} = X''Y, \quad U_{yy} = XY'' \]

\[ U_{xx} + U_{yy} = X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = \lambda \]

Also,
- \( U(0, y) = X(0) Y(y) \Rightarrow X(0) = 0 \)
- \( U(\pi, y) = X(\pi) Y(y) \Rightarrow X(\pi) = 0 \)
- \( U(x, 0) = X(x) Y(0) \Rightarrow Y(0) = 0 \)

Get 2 ODEs: (i) \( X'' = \lambda X, \quad X(0) = X(\pi) = 0 \)

(ii) \( Y'' = -\lambda Y, \quad Y(0) = 0 \).

**Step 2.** Solve for \( X \) and \( Y \)

(i) We've done this before. \( \lambda = -n^2, \quad X_n(x) = b_n \sin nx \)

(ii) \( Y'' = n^2 Y, \quad Y(0) = 0 \).

\[ Y(y) = A_n \cosh ny + B_n \sinh ny \]

\[ Y(0) = A_n = 0 \quad \Rightarrow \quad Y(y) = B_n \sinh ny \]

**Step 3:** The general soln is thus

\[ U(x, y) = \sum_{n=0}^{\infty} X_n(x) Y_n(y) = \sum_{n=1}^{\infty} b_n \sin nx \sinh ny \]
Step 4: Use the last boundary condition.

\[ u(x, y) = \sum_{n=1}^{\infty} \left( b_n \sinh n\pi \right) \sin nx = \frac{1}{2} \sin x - \frac{2}{3} \sin 2x + \frac{3}{5} \sin 3x \]

\( n = 1 \) : \( b_1 \sinh \pi = \frac{1}{2} \Rightarrow b_1 = \frac{1}{5 \sinh \pi} \)

\( n = 2 \) : \( b_2 \sinh 2\pi = -\frac{2}{3} \Rightarrow b_2 = \frac{2}{5 \sinh 2\pi} \)

\( n = 3 \) : \( b_3 \sinh 3\pi = \frac{3}{5} \Rightarrow b_3 = \frac{3}{5 \sinh 3\pi} \)

Our particular sol'n is therefore:

\[ u(x, y) = \frac{1}{5 \sinh \pi} \sin x \sinh y - \frac{2}{5 \sinh 2\pi} \sin 2x \sinh 2y + \frac{3}{5 \sinh 3\pi} \sin 3x \sinh 3y \]

Example 2: Consider a similar PDE:

\[ u_{xx} + u_{yy} = 0 \]

\( u(0, y) = 0 \), \( u(\pi, y) = y(\pi - y) \)

\( u(x, 0) = u(x, \pi) = 0 \)

Proceed as before, and get:

\[ Y'' = \lambda Y \]

\( Y(0) = Y(\pi) = 0 \Rightarrow Y_n(y) = b_n \sin ny \)

\[ X'' = \lambda X \]

\( X(0) = 0 \Rightarrow X_n(x) = B_n \sinh nx \)

General sol'n:

\[ u(x, y) = \sum_{n=1}^{\infty} b_n \sinh nx \sin ny \]

Initial cond:

\[ u(\pi, y) = y(\pi - y) = \sum_{n=1}^{\infty} \frac{y}{\pi \lambda^n} (1 - (-1)^n) \sin ny = \sum_{n=1}^{\infty} \left( b_n \sinh n\pi \right) \sin ny \]

\[ \Rightarrow b_n \sinh n\pi = \frac{4(1 - (-1)^n)}{\pi \lambda^n} \Rightarrow b_n = \frac{4(1 - (-1)^n)}{\pi \lambda^n \sinh n\pi} \]

Particular sol'n:

\[ u(x, y) = \sum_{n=1}^{\infty} \frac{4(1 - (-1)^n)}{\pi \lambda^n \sinh n\pi} \sinh nx \sin ny \]
Example 3: Consider a superposition of the 2 previous examples.

\[ u_{xx} + u_{yy} = 0 \]

\[ u(0, y) = 0, \quad u(x, 0) = 0 \]

\[ u(\pi, y) = \frac{1}{2} \sin x - \frac{2}{3} \sin 2x + \frac{3}{5} \sin 3x \]

\[ u(x, \pi) = y(\pi - y) \]

Intuitively, the solution \( u(x, y) \) (think: steady-state sol'n of these 2 PDE's; superimposed boundary conditions) should be the superposition (sum) of the 2 solutions from Example 1 & 2

i.e.,

\[ \begin{aligned}
\text{\includegraphics[width=0.3\textwidth]{image1}} + \text{\includegraphics[width=0.3\textwidth]{image2}} = \text{\includegraphics[width=0.4\textwidth]{image3}}
\end{aligned} \]

Thus, the particular sol'n is

\[ u(x, y) = \left( \frac{1}{5 \sinh \pi} \sin x \sinh y + \frac{8}{\pi \sinh \pi} \sinh x \sin y \right) \]

\[ - \left( \frac{2}{5 \sinh 2\pi} \sin 2x \sinh 2y \right) \]

\[ + \left( \frac{3}{5 \sinh 3\pi} \sin 3x \sinh 3y + \frac{8}{27 \pi \sinh 3\pi} \sinh 3x \sin 3y \right) \]

\[ + \sum_{n=4}^{\infty} \frac{4(1 - (-1)^n)}{\pi n^3 \sinh n\pi} \sin nx \sinh ny \]

Note: This can all be done more generally in a rectangular \( L \times L \) region, instead of just \( \pi \times \pi \).
Heat equation in 2D (Example): \( U_t = c^2 (U_{xx} + U_{yy}) \)

Assume \( U(x, y, t) \) = temp of a square region, \( 0 < x, y < \pi \).

\( U(0, y, t) = U(\pi, y, t) = U(x, 0, t) = U(x, \pi, t) = 0 \) (Boundary fixed at 0°).

\( U(x, y, 0) = 2 \sin x \sin 2y + 3 \sin 4x \sin 5y \) (Init. heat distribution).

Assume soln has the form \( U(x, y, t) = f(x, y) g(t) \).

Plug back in.

\( U_t = c^2 (U_{xx} + U_{yy}) \Rightarrow g'f = c^2 g f_{xx} + c^2 g f_{yy} \)

\[ \Rightarrow \frac{g'}{c^2 g} = \frac{f_{xx} + f_{yy}}{f} = \frac{\nabla^2 f}{f} = \lambda. \]

Get 2 equations:

\[ g' = c^2 \lambda g \Rightarrow g(t) = C e^{c^2 \lambda t} \]

\[ \nabla^2 f = \lambda f \] "Helmholtz equation"

Boundary conditions on \( f \):

\( U(0, y, t) = f(0, y) g(t) = 0 \Rightarrow f(0, y) = 0 \)

\( U(\pi, y, t) = f(\pi, y) g(t) = 0 \Rightarrow f(\pi, y) = 0 \)

Need to solve: \( f_{xx} + f_{yy} = \lambda f, \quad f(0, y) = f(\pi, y) = 0 \)

Assume \( f(x, y) = X(x) Y(y) \).

\( f_{xx} = X'' Y, \quad f_{yy} = X Y'' \)

Plug back in:

\[ \frac{X'' Y + X Y''}{X Y} = \frac{dXY}{X Y} = \frac{X''}{X} + \frac{Y''}{Y} = \lambda \]

\[ \Rightarrow \frac{X''}{X} = \lambda - \frac{Y''}{Y} = \mu \]

\( \mu \) depends only on \( x \), \( \lambda \) depends only on \( y \).
Get 2 ODEs:

\[ x'' = \mu x, \quad y'' = (\lambda - \mu) y \Rightarrow \lambda - \mu = \nu \Rightarrow \lambda = \nu + \mu. \]

Recall boundary conditions:

\[
\begin{align*}
F(0, y) &= x(0) y'(0) = 0 \Rightarrow x(0) = 0 \\
F(\pi, y) &= x(\pi) y'(\pi) = 0 \Rightarrow x(\pi) = 0 \\
F(x, 0) &= x(x) y'(0) = 0 \Rightarrow y'(0) = 0 \\
F(x, \pi) &= x(x) y'(\pi) = 0 \Rightarrow y'(\pi) = 0
\end{align*}
\]

ODE 1: \[ x'' = \mu x, \quad x(0) = x(\pi) = 0 \Rightarrow x_n(x) = b_n \sin nx, \quad \mu = -n^2 \]

ODE 2: \[ y'' = \nu y, \quad y(0) = y(\pi) = 0 \Rightarrow y_m(y) = B_m \sin my, \quad \nu = -m^2 \]

Recall: \[ \lambda = \nu + \mu = -(n^2 + m^2). \]

Thus, for each pair \( m \neq n \), we have a sol'n

\[ f_{nm}(x, y) = b_{nm} \sin nx \sin ny \]

Also, \( g(t) = Ce^{\lambda t} \Rightarrow g_{nm}(t) = C_{nm} e^{-c^2(n^2+m^2)t} \]

Thus, the general sol'n is

\[
\begin{aligned}
\Phi(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{nm} \sin nx \sin my e^{-c^2(n^2+m^2)t} \\
&= \sum_{n=1}^{\infty} b_{nm} \sin nx \sin my e^{-c^2(n^2+m^2)t}
\end{aligned}
\]

Use init. cond: \[ \Phi(x, y, 0) = \sum_{n=1}^{\infty} b_{nm} \sin nx \sin my \]

\[ = 2 \sin x \sin 2y + 3 \sin 4x \sin 5y \]

\[ \Rightarrow b_{12} = 2, \quad b_{45} = 3, \quad \text{all other } b_{nm} = 0. \]

Particular sol'n:

\[ \Phi(x, y, t) = 2 \sin x \sin 2y e^{-5c^2t} + 3 \sin 4x \sin 5y e^{-41c^2t} \]
Wave equation in 2D (Example): \( U_{tt} = c^2 (U_{xx} + U_{yy}) \)

Let \( u(x, y, t) = \) displacement of a square membrane of side length \( \pi \).

\[ U(0, y, t) = U(\pi, y, t) = U(x, 0, t) = U(x, \pi, t) = 0 \quad \text{(Boundary is immobile)} \]

\[ U(x, y, 0) = p(x) q(y) \quad \text{Initial wave (displacement)} \]

\[ U_t(x, y, 0) = 0 \quad \text{Initial wave velocity (vertically)} \]

Let's solve this if \( p(x) = x(\pi-x) \)

\[ q(y) = y(\pi-y) \]

Initial wave: "paraboloid-like"

Solving this is almost the same as solving the 2D heat eq'n. The only difference is \( g_{nn}(t) \)!

Assume \( u(x, y, t) = f(x, y) g(t) \).

Plug back in: \( f g'' = c^2 g f_{xx} + c^2 g f_{yy} \)

\[ \Rightarrow \frac{g''}{c^2 g} = \frac{f_{xx} + f_{yy}}{f} = \nabla^2 f / f = \lambda \]

Get 2 equations:

\[ g'' = c^2 \lambda g \quad \text{Was } g' = c^2 \lambda g \text{ in heat eq'n.} \]

\[ \nabla^2 f = \lambda F \quad \text{Same as in heat eq'n!} \]

Init cond. on \( g \): \( u_t(x, y, 0) = f(x, y) g'(0) = 0 \Rightarrow g'(0) = 0 \).
Solve for $f$: \[ \nabla^2 f = \lambda f, \quad f(0, y) = f(\pi, y) = f(x, 0) = f(x, \pi) = 0 \]

$\lambda = -(n^2 + m^2)$, \[ f_{nm}(x, y) = b_{nm} \sin nx \sin mx \]

(Same as in 2D heat equation).

Solve for $g$: \[ g'' = c^2 \lambda g, \quad g'(0) = 0 \]

$g'' = -c^2(n^2 + m^2)g$

$\Rightarrow g_{nm}(t) = A_{nm} \cos(c\sqrt{n^2+m^2}t) + B_{nm} \sin(c\sqrt{n^2+m^2}t)$

$g'(0) = 0 \Rightarrow B_{nm} = 0$

$\Rightarrow g_{nm}(t) = A_{nm} \cos(c\sqrt{n^2+m^2}t)$

For each choice of $n \in \mathbb{N}$, $m$, we have a solution to the wave equation

$U_{nm}(x, y, t) = f_{nm}(x, y) g_{nm}(t) = b_{nm} \sin nx \sin mx \cos(c\sqrt{n^2+m^2}t)$.

Thus, the general solution is

$U(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{nm} \sin nx \sin mx \cos(c\sqrt{n^2+m^2}t)$

Note: Alternatively, we could just write $\sum_{n,m \geq 1} U_{nm}(x, y, t)$.

Use initial cond.: 

\[ U(x, y, 0) = f(x, y) \]
Special topics

(1) **Laplace Transforms:**
\[ L \{ y(t) \}(s) = \int_0^\infty y(t) e^{-st} \, dt = Y(s) \]

"Turns time-derivative into multiplication by \( s \):
\[ L \{ y'(t) \}(s) = s \cdot Y(s) - y(0) \]
\[ L \{ y''(t) \}(s) = s^2 Y(s) - s y(0) - y'(0) \], and so on.

We can also take
\[ L \{ u(x, t) \}(s) = \int_0^\infty u(x, t) e^{-st} \, dt = U(x, s) \]

Similarly,
\[ L \{ u_x(x, t) \}(s) = s U(x, s) - u(x, 0) \]
\[ L \{ u_{tt}(x, t) \}(s) = s^2 U(x, s) - s u(x, 0) - u_t(x, 0) \]
\[ L \{ u_x(x, t) \}(s) = U_x(x, s) \]

*\( Y \) turns ODEs into algebraic equations,*
*PDEs into ODE (if e.g., \( u(x, t) \)).*

```
\[
\begin{align*}
\text{PDE} & \quad \xrightarrow{L} \quad \text{ODE} & \quad \text{solve} & \quad \text{ODE solution} \\
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
\text{PDE solution} & \quad \xleftarrow{L^{-1}} \quad \text{ODE solution} & \quad \text{solution} & \quad \text{solution}
\end{align*}
\]
```

(2) **Fourier Transforms:**
\[ \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} \, dx \quad \text{(or } F(\xi)) \]

"Turns special-derivative into multiplication by \( 2\pi i \xi \):
\[ \hat{f}'(\xi) = 2\pi i \xi \hat{f}(\xi) \], or more generally,
\[ \hat{f}^{(n)}(\xi) = (2\pi i \xi)^n \hat{f}(\xi) \]
Similarly, 
\[ \hat{u}_x(x,t) = (2\pi i x) \hat{u}(x,t), \quad \text{etc.} \]
\[ \hat{u}_t(x,t) = \hat{u}_c(x,t). \]

\[ \hat{u}_c(x,t) = e^{-2\pi i x} \hat{u}(x,t). \]

**Fourier transforms turn PDEs into ODEs.**

\[
\begin{array}{ccc}
\text{PDE} & \mathcal{F} & \text{ODE} \\
\downarrow & \text{solve} & \downarrow \mathcal{F}^{-1} \\
\text{PDE solution} & \text{ODE solution} & \\
\end{array}
\]

And, the inverse Fourier transform is easy! It's (almost) the same.

\[ f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi. \]

A neat way to think of Fourier (or Laplace) transforms.

Let \( f(x) \) be 2\( \pi \)-periodic. We can plot \( f \) as a function...

of \( x \) (space) \quad OR \quad of \( \xi \) (frequency).

\[ \sum_{n=-\infty}^{\infty} c_n e^{-2\pi i \xi n} \quad (\text{function of } \xi) \]

Start with \( f(x) \)

\[ f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-2\pi i x \xi} \, d\xi \]

(frequency)

Question: What would a function look like if we allowed a continuum of frequencies.

It need not even be periodic.

\[ \sum_{n=-\infty}^{\infty} c_n e^{-2\pi i n \xi} \quad (\text{function of } \xi) \]

"Frequency"
(3) Wavelets:

Big idea: Many periodic functions aren't smooth, so representing them as sines & cosines isn't natural, or efficient (for approximation purposes).

Solution: use a different basis for $\text{Per}_{2\pi}$.

e.g.,

\[
\begin{align*}
G(x) & \quad \quad \quad \quad \quad H(x) \\
-\pi & \quad \quad \quad \quad \quad \quad \pi
\end{align*}
\]

instead of

\[
\begin{align*}
\cos(x) & \quad \quad \quad \quad \quad \sin(x) \\
-\pi & \quad \quad \quad \quad \quad \quad \pi
\end{align*}
\]

Our new basis is \( \{ G(nx), H(mx), \ n \geq 0, \ m \geq 1 \} \)

instead of \( \{ \cos(nx), \sin(nx), \ n \geq 0, \ m \geq 1 \} \).

These are called Haar wavelets.

They're better to compute, e.g.,

Wavelets are used in the new & improved JPEG-2000 File Format.