MTHSC 852, Final Exam

Fall 2009

Instructor: Dr. Matthew Macauley

Wednesday, December 9, 2009

Name

Instructions

- Exam time is 2.5 hours.
- You may *not* use notes or books.
- PLEASE WRITE NEATLY AND COMPLETELY! Work out what you want to say on scratch paper before you write it out on the test. There are two pages of scratch paper at the end.
- If you have any doubt about what results you may assume, then *ask*.

Question	Points Earned	Maximum Points
1		20
2		15
3		20
4		15
5		20
6		10
Total		**100**

Student to your left:

Student to your right:

1. (a) Carefully define what it means for an element in a ring R to be (i) *prime*, and (ii) *irreducible*. Which one of these implies the other, and what is the strongest condition on R that guarantees that they are equivalent (no proof needed).

   (b) Suppose p is a prime and $f(x) = x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ with $p \mid a_i \ \ \forall i$ and $p^2 \nmid a_0$. Prove that f is irreducible over $\mathbb{Z}[x]$.

2. Let K/F be a Galois extension of degree n; let p be a prime dividing n, and write $n = p^k m$ where $p \nmid m$.

   (a) Show that there is an intermediate field E, i.e., $F \subseteq E \subseteq K$, such that $[E : F] = m$.

   (b) Show that if E/F is Galois, then E is the unique degree-m extension of F contained in K.

3. Consider the field $L = \mathbb{Q}(\sqrt[3]{1+\sqrt{2}}) \subset \mathbb{R}$.

   (a) Find generators for the Galois closure M of L over $\mathbb{Q}(\sqrt{2})$.

   (b) What are the degrees of M and L over $\mathbb{Q}$?

4. An R-module Q is *injective* if it satifies any of the following three conditions. Prove that they are equivalent.

   (i) For any unitary R-modules L, M, and N, if $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is exact, then

   $$0 \to \mathrm{Hom}_R(N,Q) \xrightarrow{g^*} \mathrm{Hom}_R(M,Q) \xrightarrow{f^*} \mathrm{Hom}_R(L,Q) \longrightarrow 0$$

   is exact.

   (ii) If $0 \longrightarrow L \xrightarrow{f} M$ is exact and $\psi \in \mathrm{Hom}_R(L,Q)$, then there exists $h \in \mathrm{Hom}_R(M,Q)$ such that $\psi = hf$.

   $$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\psi"'] & M \arrow[dl, dashed, "h"] \\ & Q & \end{tikzcd}$$

   (iii) If Q is a submodule of the R-module M, then Q is a direct summand of M, i.e., every short exact sequence $0 \longrightarrow Q \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ splits.

5. (a) Give a careful and complete definition of the following terms: (i) torsion element, (ii) the order of a torsion element, (iii) the exponent of a torsion module, (iv) the invariant factors of a torsion module, (v) the elementary divisors of a torsion module.

(b) Give a complete list of all abelian groups of order $108 = 2^2 3^3$. Classify them *both* by invariant factors and by elementary divisors.

6. What was your favorite part of the year-long sequence of MthSc 851/852, and why? What is one “big idea” that you got out of this class (in the sense that you’ll remember it prominently a year or two down the road, or that it changed your perspective). Give an example of a new connection you made between something in MthSc 851/852, to something you’ve learned in a completely different area of mathematics.