MTHSC 851/852 (Abstract Algebra)
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HW 12
Due Monday, September 7, 2009

(1) (a) Let R be a UFD (unique factorization domain, commutative), and let d a non-zero element in R. Prove that there are only finitely many principal ideals in R that contain d.
(b) Give an example of a UFD R and a nonzero element $d \in R$ such that there are infinitely many ideals in R containing d. [No proof is required for this part; however, you must describe not only R and d, but also an infinite family of ideals containing d.]

(2) Suppose $f(x) = 1 + x + x^2 + \cdots + x^{p-1}$, where $p \in \mathbb{Z}$ is prime.
(a) Show that f is irreducible in $\mathbb{Q}[x]$. [Hint: Write $f(x) = (x^p - 1)/(x-1)$, and substitute $x+1$ for x].
(b) Show that $\binom{p}{k} = \sum_{i=1}^{k+1} \binom{p-i}{p-k-1}$ for all $k < p$.

(3) (a) All of the following rings R_i, for $i = 1, \dots, 6$ are additionally $\mathbb{C}$-vector spaces. In each case, compute the vector space dimension by explicitly giving a basis for R_i over $\mathbb{C}$ in each case.
$R_1 = \mathbb{C}[x]/(x^3 - 1)$
$R_2 = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$
$R_3 =$ the ring of upper triangular 2×2 matrices over $\mathbb{C}$
$R_4 = \mathbb{C}[x]/(x-1) \times \mathbb{C}[x]/(x+i) \times \mathbb{C}[x]/(x-i)$
$R_5 = \mathbb{C}[x]/(x^2+1) \times \mathbb{C}[x]/(x-1)$
$R_6 = \mathbb{C}[x]/(x+1)^2 \times \mathbb{C}[x]/(x-1)$
(b) Partition the set $\{R_1, ..., R_6\}$ into isomorphism classes and prove your answer. [Hint: Apply the Chinese Remainder Theorem to $\mathbb{C}[x]$.]

(4) (The Euclidean Algorithm). Suppose R is a Euclidean domain, $a, b \in R$ and $ab \neq 0$. Write

$$\begin{aligned}
a &= bq_1 + r_1\,, & d(r_1) &< d(b)\,,\\
b &= r_1 q_2 + r_2\,, & d(r_2) &< d(r_1)\,,\\
r_1 &= r_2 q_3 + r_3\,, & d(r_3) &< d(r_2)\,,\\
&\vdots \\
r_{k-2} &= r_{k-1} q_k + r_k\,, & d(r_k) &< d(r_{k-1})\,.
\end{aligned}$$

with all $r_i, q_j \in R$. Show that $r_k = (a, b)$ and "solve" for r_k in terms of a and b, thereby expressing (a, b) in the form $ua + vb$ with $u, v \in R$.

(5) Use the Euclidean Algorithm to find $d = (a, b)$ and to write $d = ua + vb$ in the following cases:
(a) $a = 29041$, $b = 23843$, $R = \mathbb{Z}$;
(b) $a = x^3 - 2x^2 - 2x - 3$, $b = x^4 + 3x^3 + 3x^2 + 2x$, $R = \mathbb{Q}[x]$;
(c) $a = 7 - 3i$, $b = 5 + 3i$, $R = R_{-1}$.

(6) (a) Solve the congruences.

$$x \equiv 1 \pmod{8}\,, \qquad x \equiv 3 \pmod{7}\,, \qquad x \equiv 9 \pmod{11}$$

simultaneously for x in the ring $\mathbb{Z}$ of integers.
(b) Solve the congruences.

$$x \equiv i \pmod{i+1}\,, \qquad x \equiv 1 \pmod{2-i}\,, \qquad x \equiv 1+i \pmod{3+4i}$$

simultaneously for x in the ring R_{-1} of Gaussian integers.
(c) Solve the congruences.

$$f(x) \equiv 1 \pmod{x-1}\,, \qquad f(x) \equiv x \pmod{x^2+1}\,, \qquad f(x) \equiv x^3 \pmod{x+1}$$

simultaneously for $f(x)$ in $F[x]$, where F is a field in which $1 + 1 \neq 0$.