MTHSC 851/852 (Abstract Algebra)
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HW 14
Due Friday, October 9, 2009

(1) Prove Proposition 2.6 from lecture:
  (a) If $F \subseteq E \subseteq K$ and E is stable, then $\mathscr{G}E \lhd G$.
  (b) If $H \lhd G$, then $\mathscr{F}H$ is stable.
(2) For each field extension, compute the degree, give a basis, and find the Galois group.
  (a) $\mathbb{Q}(\sqrt[4]{2})$ over $\mathbb{Q}$
  (b) $\mathbb{Q}(\sqrt{2}, \sqrt{3}, i)$ over $\mathbb{Q}$
  (c) $\mathbb{Q}(\sqrt[3]{2}, \omega)$ over $\mathbb{Q}$, where ω is a primitive third root of unity.
  (d) $\mathbb{Q}(\omega)$ over $\mathbb{Q}$, where ω is a primitive n^{th} root of unity.
  (e) A degree-n extension of a finite field $\mathbb{F}_q$ (where $q = p^k$), over $\mathbb{F}_p$.
(3) Let $\alpha = \sqrt{3} + \sqrt[3]{2} \in \mathbb{R}$ and $K = \mathbb{Q}(\alpha)$.
  (a) Find $[K : \mathbb{Q}]$.
  (b) Let $f(x)$ be the minimal polynomial for α over $\mathbb{Q}$, and G be the Galois group of $f(x)$ over $\mathbb{Q}$. Find the order of G.
(4) Suppose that $F \subseteq K$ is a field extension of degree $n < \infty$ and E is any field containing F.
  (a) Prove that there are at most n distinct F-homomorphisms $\varphi : K \to E$ (i.e., $\varphi(x) = x$ for all $x \in F$).
  (b) Show that if E is algebraically closed, there exists at least one F-homomorphism $K \to E$.
  (c) Show that if $n = p$ is prime, then there need only exist one F-homomorphism $K \to E$.
(5) Let K/F be a Galois extension of degree 2^n, and suppose that $\text{char}(F) \neq 2$. Show that there exists a chain of intermediate subfields

$$F = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = K$$

such that $M_i = F(a_i)$, where $a_i^2 \in M_{i-1}$.
(6) Prove that $(\mathbb{Q}, +)$ is not isomorphic to the Galois group of any algebraic field extension