MTHSC 851/852 (Abstract Algebra)
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HW 15
Due Monday, October 19, 2009

(1) (a) Find a primitive element over $\mathbb{Q}$ for $K = \mathbb{Q}(\sqrt{3}, \sqrt[3]{2}) \subseteq \mathbb{C}$.
  (b) Find a primitive element over $\mathbb{Q}$ for a splitting field $K \subseteq \mathbb{C}$ for the polynomial $f(x) = x^4 - 5x^2 + 6$.

(2) Let K/F be a normal field extension and $f(x) \in F[x]$ an irreducible polynomial over F.
  (a) Prove that if $f(x)$ splits in K, all zeros of $f(x)$ in K have the same multiplicity.
  (a) Now suppose that $f(x)$ does not split in K. Prove that all irreducible factors of $f(x)$ in $K[x]$ have the same degree.

(3) (a) Let F be a field of characteristic zero, and let p be a prime such that $p \mid [K : F]$ for every field extension K/F of finite degree. Prove that $[K : F]$ is a power of p whenever K/F is an extension of finite degree.
  (b) Let F be a field, $\mathbb{Q} \subseteq F \subseteq \mathbb{A}$, maximal with respect to $\sqrt{2} \notin F$ (Why does F exist?).
    (i) If $F \subseteq K \subseteq \mathbb{A}$, with K normal and finite over F, and $K \neq F$, show that $G = \text{Gal}(K/F)$ is a 2-group having a unique subgroup of index 2. Conclude that G is cyclic.
    (ii) If $F \subseteq L \subseteq \mathbb{A}$ and $[L : F]$ is finite show that L is normal over F and $\text{Gal}(L/F)$ is cyclic. Conclude that the set of finite extensions of F (in $\mathbb{A}$) is an ascending chain.

(4) Suppose K/F has finite degree and $\text{char}\, F \nmid [K : F]$. Show that K/F is separable.

(5) Let F be a field of characteristic p and $f(x) = x^p - a \in F[x]$. Prove that $f(x)$ is either irreducible in $F[x]$ or splits in $F[x]$.

(6) Suppose $\text{char}\, F = p \neq 0$ and K is an extension of F. An element $a \in K$ is called purely inseparable over F if it is a root of a polynomial of the form $x^{p^k} - b \in F[x]$, $0 \leq k \in \mathbb{Z}$.
  (a) Show that if $a \in K$ is both separable and purely inseparable over F, then $a \in F$.
  (b) Show that the set of all elements of K that are purely inseparable over F constitute a field. Conclude that there is a unique largest "purely inseparable" extension of F within K.