MTHSC 851/852 (Abstract Algebra)
Dr. Matthew Macauley
HW 16
Due Friday, November 1st, 2009

(1) Suppose $F \subseteq E \subseteq K$, $F \subseteq L \subseteq K$, $\mathrm{Gal} = G(K/F)$, $J \leq G$, and $H \leq G$
  (a) Show that $\mathscr{G}(E \vee L) = \mathscr{G}E \cap \mathscr{G}L$ and $\mathscr{F}(J \vee H) = \mathscr{F}J \cap \mathscr{F}H$.
  (b) Show that $[E \vee L : F] \leq [E : F][L : F]$.

(2) A field F is called perfect if either $\operatorname{char} F = 0$ or else $\operatorname{char} F = p$ and $F = F^p = \{a^p : a \in F\}$.
  (a) If F is finite show that the map $a \mapsto a^p$ is a monomorphism and conclude that F is perfect.
  (b) Show that the field $\mathbb{Z}_p(t)$ of rational functions in the indeterminate t is not perfect.
  (c) Show that a field F is perfect if and only if every finite extension K of F is separable over F, and hence every $f(x) \in F[x]$ is separable.

(3) Let F be any infinite field and $F(x)$ a simple transcendental extension. Prove that $F \subseteq F(x)$ is a Galois extension.

(4) If $S \subseteq K$ and K is algebraic over $F(S)$ show that there is a transcendence basis B for K over F with $B \subseteq S$

(5) (a) Let $G = \mathrm{Gal}(\mathbb{R}/\mathbb{Q})$. If $\phi \in G$ and $a \leq b$ in $\mathbb{R}$ show that $\phi(a) \leq \phi(b)$. [Hint: $b - a$ is a square in $\mathbb{R}$.]
  (b) Show that $G = 1$. [Hint: If not choose $\phi \in G$ and $a \in \mathbb{R}$ such that $\phi(a) \neq a$. Choose $b \in \mathbb{Q}$ between a and $\phi(a)$.]

(6) Let $F \subseteq K$ be a field extension.
  (a) Suppose $K = F(x)$ is simple transcendental, and show that there are infinitely many intermediate fields $F \subseteq L \subseteq K$.
  (b) Prove the same conclusion as (a) whenever $[K : F]$ is infinite.