MTHSC 851/852 (Abstract Algebra)
Dr. Matthew Macauley
HW 17
Due Monday, November 16th, 2009

(1) If R is a ring with 1 and M is an R-module that is not unitary, show that $Rm = 0$ for some $m \neq 0$.

(2) If F is a field set $R = F[x_1, x_2, x_3, \dots]$, the ring of polynomials in a countably infinite set of distinct indeterminantes. Let I be the ideal $(x_1, x_2, \dots)$ in R. If $M = R$ and $N = I$ show that M is a finitely generated R-module but N is a submodule that is not finitely generated. Is N free?

(3) Suppose L, M and N are R-modules and $f : M \to N$ is an R-homomorphism. Define $f^* : \mathrm{Hom}_R(N, L) \to \mathrm{Hom}_R(M, L)$ via $f^*(\phi) : m \mapsto \phi(f(m))$ for all $\phi \in \mathrm{Hom}_R(N, L)$, $m \in M$.

(a) Show that f^* is a $\mathbb{Z}$-homomorphism.

(b) If R is commutative show that f^* is an R-homomorphism.

(c) Still assuming that R is commutative, show that if $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an exact sequence of R-modules, then for any R-module D, the sequence

$$0 \to \mathrm{Hom}_R(N, D) \xrightarrow{g^*} \mathrm{Hom}_R(M, D) \xrightarrow{f^*} \mathrm{Hom}_R(L, D)$$

is an exact sequence of abelian groups.

(4) The *Five Lemma* states that given a diagram of abelian groups

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ A_1' & \longrightarrow & A_2' & \longrightarrow & A_3' & \longrightarrow & A_4' & \longrightarrow & A_5' \end{array}$$

where the rows are exact, and f_1, f_2, f_4 and f_5 are isomorphisms, f_3 is an isomorphism as well.

(a) Prove the Five Lemma.

(b) Consider the following eight hypotheses:

$$f_i \text{ is injective, for } i = 1, 2, 4, 5,$$
$$f_i \text{ is surjective, for } i = 1, 2, 4, 5.$$

Which of these hypothese suffice to prove that f_3 is injective? Which suffice to prove that f_3 is surjective?