MTHSC 851, Final Exam

Spring 2009

Instructor: Dr. Matthew Macauley

Monday, April 27, 2009

Name

Instructions

- Exam time is 2.5 hours.
- You may *not* use notes or books.
- PLEASE WRITE NEATLY AND COMPLETELY! Work out what you want to say on scratch paper before you write it out on the test. There are two pages of scratch paper at the end.
- If you have any doubt about what results you may assume, then *ask*.

Question	Points Earned	Maximum Points
1		10
2		15
3		20
4		20
5		15
6		20
Total		**100**

Student to your left:

Student to your right:

1. (a) Describe in full the meaning of the statement that a group G has a presentation

$$\langle a, b \mid a^2 = b^3 = 1,\ ab^{-1}ab = b^{-1}aba \rangle.$$

(b) Carefully state the Fundamental Theorem of Finite Abelian Groups.

2. (a) Carefully and fully define the following terms (listed in alphabetical order): (i) Commutative Ring, (ii) Euclidean Domain, (iii) Field, (iv) Integral Domain, (v) Principal Ideal Domain.

(b) Order the above five classes of rings by size, e.g., by filling in the following:

$$\underline{\qquad} \subsetneq \underline{\qquad} \subsetneq \underline{\qquad} \subsetneq \underline{\qquad} \subsetneq \underline{\qquad}.$$

(c) For each proper inclusion, give an explicit example of a ring that is in the larger class but not in the smaller class.

3. (a) Let p be a prime, G a finite p-group acting on a finite set X, and n the number of fixed points for the action, i.e., the cardinality of the set $\{x \in X \mid gx = x,\ \forall g \in G\}$. Show that $n \equiv |X| \pmod{p}$.

(b) Use part (a) – instead of the Class Equation – to prove that the center of any nontrivial finite p-group is nontrivial.

4. (a) Show that S_3 has a presentation $\langle a, b \mid a^2 = b^3 = 1,\ aba = b^{-1} \rangle$.

(b) Use part (a) to show that every non-abelian group of order 6 is isomorphic to S_3. [Hint: Subgroups of index 2 are normal.]

5. Let A_1, A_2, A be objects in a category $\mathfrak{C}$, and let $f_i \in \mathrm{Hom}_{\mathfrak{C}}(A_i, A)$ for $i = 1, 2$. A *pullback* (or *fiber product*) for the 5-tuple (A_1, A_2, A, f_1, f_2) is a commutative diagram

$$\begin{tikzcd} B \arrow[r, "g_1"] \arrow[d, "g_2"'] & A_1 \arrow[d, "f_1"] \\ A_2 \arrow[r, "f_2"] & A \end{tikzcd}$$

of objects and morphisms in $\mathfrak{C}$ such that the following property holds:

> For any object C in $\mathfrak{C}$ and any morphisms $h_i \in \mathrm{Hom}_{\mathfrak{C}}(C, A_i)$ such that if $f_1 h_1 = f_2 h_2$, there exists a unique morphism $h \in \mathrm{Hom}_{\mathfrak{C}}(C, B)$ such that $h_i = g_i h$ for $i = 1, 2$.

Now suppose that

$$\begin{tikzcd} B \arrow[r, "g_1'"] \arrow[d, "g_2'"'] & A_1 \arrow[d, "f_1"] \\ A_2 \arrow[r, "f_2"] & A \end{tikzcd}$$

is another pullback for (A_1, A_2, A, f_1, f_2). Show that $B' \cong B$.

6. (a) Suppose R is a commutative ring with 1 and $x \in \cap \{M \mid M \text{ is a maximal ideal in } R\}$. Show that $1 + x \in U(R)$. [*Hint*: Use the fact that every non-unit is contained in a maximal ideal.]

(b) Suppose R is a commutative ring, I and J are ideals in R, P is a prime ideal in R, and $I \cap J \subseteq P$. Show that $I \subseteq P$ or $J \subseteq P$.