MTHSC 851 (Abstract Algebra)
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HW 1
Due Tuesday Jan. 20, 2009

(1) Let G be a finite group and $n > 2$. Show that the number of elements in G of order n is even.
(2) Show that every group of even order has an element of order 2.
(3) If a group G has a unique element x of order 2 show that $x \in Z(G)$.
  (a) Show that any finitely-generated subgroup of $(\mathbb{Q}, +)$ (the additive group of the rationals) is cyclic. Use this to show that $(\mathbb{Q}, +)$ is not isomorphic to $(\mathbb{Q}, +) \oplus (\mathbb{Q}, +)$ (the direct sum of two copies of $\mathbb{Q}$.)
  (b) What happens if $\mathbb{Q}$ is replaced by $\mathbb{R}$ in both parts of (a)?
(4) Suppose G is a group, $H \leq G$, and $K \leq G$. Give necessary and sufficient conditions for $H \cup K$ to be a group.
(5) Show that a group G is the union of three proper subgroups if and only if there is an epimorphism from G to Klein's 4-group.
(6) (a) Suppose $H \leq G$. Show that gHg^{-1} is a subgroup, and that $H \cong gHg^{-1}$.
  (b) Use (a) to show that in any group, $|xy| = |yx|$.
(7) (a) Prove that if $G/Z(G)$ is cyclic, then G is abelian
  (b) Prove that if $Z(G)$ is maximal among abelian subgroups, then G is abelian.
(8) Let $|G| = \infty$, and $[G : H] < \infty$. Show that H intersects every infinite subgroup of G nontrivially.
(9) Suppose G is finite, $H \leq G$, and $G = \cup\{xHx^{-1} \mid x \in G\}$. Show that $H = G$.
(10) Prove or give a counter example to each statement:
  (a) If every proper subgroup H of a group G is cyclic, then G is cyclic.
  (b) If H is a subgroup of an abelian group G, then both H and the quotient group G/H are abelian.
  (c) If H is a normal abelian subgroup of a group G, and the quotient group G/H is also abelian, then G is abelian.
  (d) If $K \lhd H \lhd G$, then $K \lhd G$.