(1) Suppose \( G \) is finite, \( p \) is the smallest prime dividing \(|G|\), \( H \leq G \), and \([G : H] = p\). Show that \( H \triangleleft G \).

(2) Suppose \([G : H]\) is finite. Show that there is a normal subgroup \( K \) of \( G \) with \( K \leq H \), such that \([G : K]\) is finite.

(3) Suppose \( H \leq S_n \), but \( H \not\leq A_n \). Show that \([H : A_n \cap H] = 2\).

(4) Prove that if \( H \) and \( K \) are normal subgroups of a group \( G \) and \( HK = G \) then

\[
G/(H \cap K) \cong (G/H) \times (G/K).
\]

(5) Prove the tower law: If \( K \leq H \leq G \), then \([G : K] = [G : H][H : K]\).

(6) Suppose \( G \) is finite, \( H \leq G \), \([G : H] = n\), and \(|G| \not\mid n!\). Show that there is a normal subgroup \( K \) of \( G \), \( K \neq 1 \), such that \( K \leq H \).

(7) If \(|G| = p^n\) for some prime \( p \), and \( 1 \neq H \triangleleft G \), show that \( H \cap Z(G) \neq 1 \).

(8) If \( A, B \leq G \) and \( y \in G \) define \((A, B)\)-double coset \( AyB = \{ayb \mid a \in A, b \in B\}\). Show that \( G \) is the disjoint union of its \((A, B)\)-double cosets. Show that \(|AyB| = [A^y : A^y \cap B] \cdot |B|\) if \( A \) and \( B \) are finite.

(9) Let \( G \) be a group of order 15, which acts on a set \( S \) with 7 elements. Show the group action has a fixed point.

(10) Suppose \( G \) acts on \( S \), \( x \in G \), and \( x \in S \). Show that \( \text{Stab}_G(xs) = x \text{Stab}_G(s)x^{-1} \).

(11) Prove that if \( G \) contains no subgroup of index 2, then any subgroup of index 3 is normal in \( G \).

(12) Suppose that \( H \) and \( K \) both have finite index in \( G \). Prove that \([G : H \cap K] \leq [G : H][G : K]\).