MTHSC 851 (Abstract Algebra)
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HW 3
Due Thursday Feb. 5, 2009

(1) Let A and B be finite groups of G. Even though AB need not be a subgroup of G, show that $|AB| \cdot |A \cap B| = |A| \cdot |B|$. (*Hint*: define $(a_1, b_1) \sim (a_2, b_2)$ iff $a_1 b_1 = a_2 b_2$. Prove that $\sim$ is an equivalence relation and examine the equivalence classes.)
(2) If $A \lhd G$ and $B \lhd G$ show that $G/(A \cap B)$ is isomorphic to a subgroup of $G/A \times G/B$.
(3) Let G be a non-cyclic finite p-group. Show that there is a epimorphism $G \longrightarrow \mathbb{Z}_p \times \mathbb{Z}_p$.
(4) (a) Write out the conjugacy classes explicity in S_3 and S_4.
  (b) What are the conjugacy classes in A_4?
  (c) Since $|A_4| = 12$, any subgroup of order 6 would be normal. Use (b) to show that A_4 has no subgroup of order 6. Conclude that the converse to Lagrange's Theorem is false.
  (d) Find a normal subgroup of order 4 in A_4.
(5) If $|G| = p^n$, p a prime, show that G has subgroups $G_0, G_1, \dots, G_n$ with $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ such that $[G_i \colon G_{i-1}] = p$, $1 \leq i \leq n$.
(6) (a) How many subgroups does S_4 have isomorphic to S_3?
  (b) How many subgroups does S_4 have isomorphic to S_2?
(7) Let G be a group, not necessarily finite, and let $H \leq G$.
  (a) Prove that $U = \cap_{x \in G} x H x^{-1}$ is the largest normal subgroup of G contained in H.
  (b) Show that no proper subgroup H of A_5 contains six distinct Sylow 5-subgroups.
(8) If P is a p-Sylow subgroup of G, show tha $N_G(N_G(P)) = N_G(P)$.
(9) (a) Show that if $|G| = pq$, where p and q are prime, then G is not simple.
  (b) Show that the only simple groups of order less than 36 are of prime order.
(10) Let G be a simple group of order 168. Show that G is a subgroup of A_8, the alternating group.
(11) Let G be a group of order 108.
  (a) Prove that there exists a nontrivial homomorphism $G \to S_4$.
  (b) Show that G is not simple.
(12) Let G be a group of order 90, and assume that G has no normal Sylow 5-subgroups.
  (a) Show that there is a nontrivial homomorphism $\phi : G \to S_6$.
  (b) If $\phi(G) \subseteq A_6$, show that ϕ is not injective.
  (c) Show that G is not simple.