MTHSC 851 (Abstract Algebra)
Dr. Matthew Macauley
HW 4
Due Friday Feb. 13, 2009

(1) Suppose G is a finite group, $H \lhd G$, and P is a p-Sylow subgroup of H. Set $N = N_G(P)$. Show that $G = NH$.

(2) Permutation groups G_1 and G_2 acting on sets S_1 and S_2 are called *permutation isomorphic* if there exist an isomorphism $\theta\colon G_1 \to G_2$ and a bijection $\phi\colon S_1 \to S_2$ such that $(\theta x)(\phi s) = \phi(xs)$ for all $x \in G_1$ and $s \in S_1$. In other words, the following diagram commutes:

$$\begin{array}{ccc} S_1 & \xrightarrow{x} & S_1 \\ \phi\downarrow & & \downarrow\phi \\ S_2 & \xrightarrow[\theta x]{} & S_2 \end{array}$$

Define two group actions of a group G on itself as follows:
  (i) the action of $x \in G$ is left multiplication by x;
  (ii) the action of $x \in G$ is right muliplication by x^{-1}.

Show that the two actions are permutation isomorphic.

(3) For each of the following statements, prove or give a counterexample.
  (i) Let $f\colon G \to H$ be an epimorphism. Then for any two homomorphsims $g_1, g_2\colon H \to K$, the equality $g_1 \circ f = g_2 \circ f$ implies that $g_1 = g_2$.
  (ii) Let $f\colon G \to H$ be a monomorphism. Then for any two homomorphsims $g_1, g_2\colon H \to K$, the equality $g_1 \circ f = g_2 \circ f$ implies that $g_1 = g_2$.
  (iii) Let $g\colon H \to K$ be an epimorphism. Then for any two homomorphsims $f_1, f_2\colon G \to H$, the equality $g \circ f_1 = g \circ f_2$ implies that $f_1 = f_2$.
  (iv) Let $g\colon H \to K$ be a monomorphism. Then for any two homomorphsims $f_1, f_2\colon G \to H$, the equality $g \circ f_1 = g \circ f_2$ implies that $f_1 = f_2$.

(4) If (U, ε) is a universal pair for a group G and $h \in \mathrm{Aut}(U)$ show that $(U, h\varepsilon)$ is also universal for G. Conversely, if (U, ε_1) is universal for G show that $\varepsilon_1 = h\varepsilon$ for some $h \in \mathrm{Aut}(U)$.

(5) Find G' if $G = S_3$, S_4, or A_4.

(6) Prove the lemma from class:
  (i) If $G' \le H \le G$ show that $H \lhd G$.
  (ii) Show that if $K \lhd G$, then $K' \lhd G$.
  (iii) Suppose $f\colon G \to H$ is an epimorphism, with $\ker f = K$. Show that H is abelian if and only if $G' \le K$.

(7) (a) Find the derived series for S_4.
  (b) Show that $S_n' = A_n$ if $n \neq 2$. Conclude that S_n is not solvable if $n \ge 5$.

(8) Show that any finite p-group is solvable.

(9) If $|G| = p^2 q$ for primes p and q, show that G is solvable.