MTHSC 851 (Abstract Algebra)
Dr. Matthew Macauley
HW 9
Due Tuesday April 14th, 2009

(1) Give an example of a ring with exactly 851 ideals.
(2) If F is a field, show that $M_n(F)$ is a simple ring.
(3) Let R be a ring with unity and $x \in R$ any non-unit. Use Zorn's lemma to prove that x is contained in a maximal ideal.
(4) A *local ring* is a commutative ring with identity which has a unique maximal ideal. Prove that R is local if and only if the non-units of R form an ideal.
(5) Let R be a finite ring.
  (a) Prove that there are positive integers m and n with $m > n$ such that $x^m = x^n$ for every $x \in R$. (*Hint*: If $|R| = n$, then consider the ring $S = R \times \cdots R$, with n factors.)
  (b) Give a direct proof (i.e., without appealing to part (c)) that if R is an integral domain, then it is a field.
  (c) Suppose that R has identity. Prove that if $x \in R$ is not a zero divisor, then it is a unit.
(6) (a) An element a of a ring R is called nilpotent if $a^n = 0$ for some positive integer n. Show that the set of nilpotent elements in a commutative ring R is an ideal of R.
  (b) If $u \in R$ is a unit and $a \in R$ nilpotent, show that $u + a$ is a unit.
(7) Let R be a commutative ring.
  (a) Show that an ideal P in R is prime if and only if R/P is an integral domain.
  (b) If additionally, R has 1, show that every maximal ideal is prime.
  (c) Give an example of an integral domain R and a nonzero prime ideal P that is not maximal.
(8) (a) If R is a field, show that R itself is a field of fractions for R.
  (b) Show that $\mathbb{Q}$ is a field of fractions for $\mathbb{Z}$ and for $2\mathbb{Z}$.
(9) Let R be any commutative ring and S a subset of $R \setminus \{0\}$ that is a semigroup under multiplication, and contains no zero devisors. Let X be the Cartesian product $R \times S$ and define a relation $\sim$ on X where $(a, b) \sim (c, d)$ if $ad = bc$.
  (a) Show that $\sim$ is an equivalence relation on X.
  (b) Denote the equivalence class of (a, b) by a/b and the set of equivalence classes by R_S (called the *localization* of R at S). Show that R_S is a commutative ring with 1.
  (c) If $a \in S$ show that $\{ra/a \colon r \in R\}$ is a subring of R_S and that $r \mapsto ra/a$ is a monomorphism, so that R can be identified with a subring with R_S.
  (d) Show that every $s \in S$ is a unit in R_S.
  (e) Give a "universal" definition for the ring R_S and show that R_S is unique up to isomorphism.