If $G_1$, $G_2$ are groups, then $G_1 \times G_2$ is a group, where the binary operation is $(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_2)$.

This notion can easily be extended to an arbitrary finite product $G_1 \times G_2 \times \ldots \times G_k$, or even a countable product $G_1 \times G_2 \times \ldots$.

But what if we have uncountably many (or more) groups?

Big idea: Everything works if we define products as (co)universal properties.

**Def.** If $\{G_x : x \in A\}$ is a non-empty family of groups, then a **product** of the $G_x$'s is a group $P$ with a family of homomorphisms $p_x : P \to G_x$, $x \in A$, with the following universal property:

Given any group $H$, and homomorphisms $f_x : H \to G_x$, $x \in A$, $\exists! \; f : H \to P \quad \text{s.t.} \quad p_x f = f_x \quad \forall x \in A$.

```
\begin{tikzcd}
H \arrow{r}{f_x} & G_x \\
& P \\
\end{tikzcd}
```
**Prop. 6.1:** Let \( \{G_a : a \in A\} \) be a non-empty family of groups. If a product \( (P, \{p_a\}) \) exists, it is unique up to isomorphism, and each \( p_a : P \to G_a \) is an epi.

**PF:** Let \( (P', \{p'_a\}) \) be another product. We have

\[
\begin{array}{ccc}
P' & \xrightarrow{p'_a} & G_a \\
\downarrow{p_a} & \downarrow{p_a'} & \downarrow{f} \\
P & \xrightarrow{p_a} & G_a
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
P & \xrightarrow{p_a} & G_a \\
\downarrow{p'_a} & \downarrow{g} & \downarrow{p'_a} \\
P' & \xrightarrow{p'_a} & G_a
\end{array}
\]

for each \( a \in A \). Thus, \( p_a = p'_a g = p_a f g \) \( \forall a \in A \).

Now, we have

\[
\begin{array}{ccc}
P & \xrightarrow{p_a} & G_a \\
\downarrow{f g} & \downarrow{p_a} & \downarrow{1} \\
P & \xrightarrow{p_a} & G_a
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
P & \xrightarrow{p_a} & G_a \\
\downarrow{1} & \downarrow{p_a} & \downarrow{p_a} \\
P & \xrightarrow{p_a} & G_a
\end{array}
\]

thus, \( f g = 1 \) (by uniqueness). Similarly, \( g f = 1 \), so \( f \) and \( g \) are inverse isomorphisms.

Now, we'll show each \( p_a \) is an epi.

Fix \( G_a \), define \( f_a : G_a \to G_a \),

\[
f_a(x) = \begin{cases} 
1 \in G_a & \text{if } x \in G_a \neq G_a \\
x \in G_a & \text{if } x \in G_a = G_a \end{cases}
\]

(trivial)

(identity)
Now, we have, for all \( \alpha \),

\[
\begin{array}{c}
G_x \\ F \\
\downarrow \ \\
\rho \\
\end{array}
\xrightarrow{\beta} 
\begin{array}{c}
G_x' \\ F' \\
\downarrow \ \\
\rho' \\
\end{array}
\quad \text{and} \quad 
\begin{array}{c}
G_x \\ F \\
\downarrow \ \\
\rho \\
\end{array}
\xrightarrow{f_x = 1_x} 
\begin{array}{c}
G_x' \\ F' \\
\downarrow \ \\
\rho' \\
\end{array}
\]

Since \( 1_x = \rho \circ f \) and \( 1_x \) is an epi, so is \( \rho_x \).

**Theorem 6.2:** If \( \{G_x : \alpha \in A\} \) is a nonempty family of groups, then the product of \( \{G_x : \alpha \in A\} \) exists.

**Proof:** We'll show that the cartesian product is the product.

Let \( \mathcal{P} = \prod_{\alpha \in A} G_x \). Write elts as \( (x_\alpha)_{\alpha \in A} \).

Binary operation: \( (x_\alpha)(y_\alpha) = (x_\alpha \cdot y_\alpha) \). This is a group.

Define \( \rho_x : \mathcal{P} \to G_x \) as the projection map \( (x_\alpha) \mapsto x_\alpha \).

Suppose \( f_x : H \to G_x \) is a homomorphism (\( \forall \alpha \) in \( A \)).

Define \( f : H \to \mathcal{P} \), \( f(h) = (f_x(h))_{\alpha \in A} \).

Check: \( \rho_x f = f_x \ \forall \alpha \in A \).

Uniqueness: Consider \( \exists \) \( f \)

\[
\begin{array}{c}
H \\ f \\
\downarrow \ \\
\rho \\
\end{array}
\xrightarrow{f_x} 
\begin{array}{c}
G_x \\ P_x \\
\downarrow \ \\
\rho \ \\
\end{array}
\quad \text{and} \quad 
\begin{array}{c}
H \\ f \\
\downarrow \ \\
\rho \\
\end{array}
\xrightarrow{f_x} 
\begin{array}{c}
G_x \\ P_x \\
\downarrow \ \\
\rho \ \\
\end{array}
\]

i.e., \( \rho_x f = \rho_x g \). Then \( f(x)_\alpha = \rho_x f(x) = \rho_x g(x) = g(x)_\alpha \) implies \( f(x) = g(x) \ \forall x \in H \). \( \square \)
Thus, \( f(x) = g(x) \neq x \in H \), and so \( f \) is unique, and \( P \) is a product. \( \square \)

If \( A = \{1, 2, \ldots, n\} \) or \( \{1, 2, 3, \ldots\} \), we write \( G_1 \times G_2 \times \cdots \times G_n \)
or \( G_1 \times G_2 \times G_3 \times \cdots \). This is also called the direct product. The homomorphism \( \pi_x \) is called the projection of \( \Pi G_x \) on the direct factor \( G_x \).

**Thm 6.3:** Suppose \( G_1, G_2 \leq G \) satisfying:

1. \( G_1, G_2 \leq G \)
2. \( G_1 \cap G_2 = 1 \)
3. \( G_1 G_2 = G \).

Then \( G \cong G_1 \times G_2 \).

More generally, if \( G_1, \ldots, G_n \leq G \) satisfying:

1. \( G_1, \ldots, G_n \leq G \)
2. \( G_i \cap \left( \bigcup_{j \neq i} G_j \right) = 1 \) for each \( i \)
3. \( G_1 G_2 \cdots G_n = G \),

then \( G \cong G_1 \times G_2 \times \cdots \times G_n \).

**Pf (Sketch, for \( n=2 \)).**

Since \( G_1 \cap G_2 = 1 \), each \( x = x_1 x_2 \) uniquely, where \( x_i \in G_i \).

Also, \([x_1, x_2] = x_1 x_2 x_1^{-1} x_2^{-1} \in G_1 \cap G_2 = 1 \) (by normality), thus \( x_1 x_2 = x_2 x_1 \).

Define our family of homomorphisms by \( \rho_i(x) = x_i \) (check homom!)

Now, for any family of homoms \( f_j : H \rightarrow G_i \),
Define \( F: H \to G \) by \( F(h) = f_1(h) f_2(h) \).

Then \( p_i F = F_i \) (check!)

If \( g: H \to G \) is a homomorphism satisfying \( p_i g = F_i \) (for each \( i \)),
then \( p_i F = p_i g \). Then, for any \( x \in H \), we have
\[
F(x) = p_1(F(x)) p_2(F(x)) = p_1(g(x)) p_2(g(x)) = g(x),
\]
thus \( F = g \) is unique.

Since products exist and are unique, and \( G \) is a product,
it follows that \( G \cong G_1 \times G_2 \).

\( \square \)

When the assumption of Thm 6.3 hold, we say that
\( G \) is the internal direct product of its subgroups
\( G_1, \ldots, G_n \).

---

Some basic category theory.

**Def:** A category \( C \) consists of

(I) A class of objects \( \text{Ob}(C) \)

(II) A class of morphisms \( \text{Hom}(C) \) between objects, with

(i) Identity morphism \( 1_A: A \to A \) for all \( A \in \text{Ob}(C) \)

(ii) Composition: \( f: \text{Hom}_C(A, B), \ g: \text{Hom}_C(B, C) \)
\[
\Rightarrow g \circ f : \text{Hom}_C(A, C),
\]

(iii) Associative: \( h \circ (g \circ f) = (h \circ g) \circ f \).
Think of a category as a directed graph

**Vertex** ↔ objects

**Edge** ↔ morphisms.

**Example**

<table>
<thead>
<tr>
<th>Set</th>
<th>Groups with homomorphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grp</td>
<td><strong>Ab</strong> Abelian groups with homomorphisms</td>
</tr>
<tr>
<td>Vect</td>
<td>K-vector spaces with K-linear maps</td>
</tr>
<tr>
<td>Top</td>
<td>Topological spaces with continuous functions</td>
</tr>
</tbody>
</table>

**Def:** A morphism \( f \in \text{Hom}_C(A, B) \) is a

- **Monomorphism** if \( f g_1 = f g_2 \Rightarrow g_1 = g_2 \)
- **Epimorphism** if \( g_1 f = g_2 f \Rightarrow g_1 = g_2 \)
- **Isomorphism** if \( \exists g \in \text{Hom}_C(B, A) \) with \( g f = 1_B \) and \( f g = 1_A \).

In this case, we say that \( A \) and \( B \) are equivalent.

**Def:** Let \( C \) be a category and \( \{ A_i : i \in I \} \) a family of objects of \( C \). A **product** for \( \{ A_i : i \in I \} \) is an object \( P \) of \( C \) with a family of morphisms \( \{ p_i : P \to A_i : i \in I \} \) such that for any object \( B \) and family of morphisms \( \{ f_i : B \to A_i : i \in I \} \),\( \exists ! f : B \to \prod_{i \in I} A_i \) s.t. \( p_i f = f_i \) \( \forall i \in I \).

Denote this as \( P = \prod_{i \in I} A_i \).
Def: A coproduct (or sum) for \( \{A_i \mid i \in I\} \) is an object \( S \in \text{Ob}(\mathcal{C}) \) with a family of morphisms \( \{L_i : A_i \rightarrow S \mid i \in I\} \) such that for any object \( B \in \text{Ob}(\mathcal{C}) \) and family of morphisms \( \{f_i : A_i \rightarrow B \mid i \in I\} \),

\[ \exists f \in \text{Hom}_\mathcal{C}(S, B) \text{ such that } fL_i = f_i \quad \forall i \in I \]

Denote this by \( S = \bigsqcup_{i \in I} X_i \), or \( S = \bigoplus_{i \in I} X_i \).

\[
\begin{align*}
\text{Product of } X_i \in X_2 & \quad \text{Coproduct of } X_i \oplus X_2 \n
\end{align*}
\]

**Examples:**

<table>
<thead>
<tr>
<th>Category</th>
<th>Objects</th>
<th>Morphisms</th>
<th>Product</th>
<th>Coproduct</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set</td>
<td>Sets</td>
<td>Functions</td>
<td>Cartesian prod</td>
<td>Disjoint union</td>
</tr>
<tr>
<td>Top</td>
<td>Top. space</td>
<td>Cont. maps</td>
<td>Cartesian prod</td>
<td>Disjoint union</td>
</tr>
<tr>
<td>Grp</td>
<td>Groups</td>
<td>Homomorphisms</td>
<td>Direct product</td>
<td>Free product</td>
</tr>
<tr>
<td>Ab</td>
<td>Abeliangps</td>
<td>Homomorphisms</td>
<td>Direct product</td>
<td>Direct sum</td>
</tr>
</tbody>
</table>
Products and coproducts are defined via universal mapping properties, i.e., by the existence of certain uniquely determined morphisms. This notion can be generalized.

**Def:**
- An object $I \in \text{Ob}(C)$ is **universal** (or initial) if for each $C_i \in \text{Ob}(C)$, $\exists ! p_i \in \text{Hom}_C(I, C_i)$.
- An object $T \in \text{Ob}(C)$ is **couniversal** (or terminal) if for each $C_i \in \text{Ob}(C)$, $\exists ! l_i \in \text{Hom}_C(C_i, T)$.
- An universal and initial object is a **zero object**.

<table>
<thead>
<tr>
<th>Examples</th>
<th>Category</th>
<th>Universal (initial)</th>
<th>Couniversal (terminal)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>${x \times x \mid \text{any } x}$</td>
</tr>
<tr>
<td>Top</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>${x \times x}$</td>
</tr>
<tr>
<td>Grp</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

**Thm 6.9:** Any two universal objects are equivalent.

**Pf.** (sketch) let $I$ and $J$ be universal.

\[
\begin{array}{ccc}
I \xrightarrow{g \circ f} I & \xrightarrow{1_I} & I \\
\downarrow f & & \downarrow g \\
I & & I \\
\end{array}
\quad
\begin{array}{ccc}
J \xrightarrow{f \circ g} J & \xrightarrow{1_J} & J \\
\downarrow g & & \downarrow f \\
I & & J \\
\end{array}
\]

Then $f \circ g = 1_J$ and $g \circ f = 1_I \Rightarrow I$ & $J$ are equivalent \(\square\)

**Note:** Similarly, we can show that any two couniversal objects are equivalent.
For suitably chosen categories, (co)products are the (co)universal objects.

**Example** Let \( \{ A_i \mid i \in I \} \) be a family of objects in \( C \).

Define a new category \( D \) as follows:

- **Objects:** Pairs \( (B, \{ F_i \mid i \in I \}) \) where \( F_i \in \text{Hom}_C(B, A_i) \).
- **Morphisms:** Elements \( h \in \text{Hom}_C(B, D) \) s.t. \( g_i \circ h = F_i \) \( \forall i \in I \).

**Check:** In this category, the couniversal (terminal) object is \( \{ \prod_{i \in I} A_i, \{ p_i \mid i \in I \} \} \) (if \( \prod_{i \in I} A_i \) exists in \( C \)).

**Cor:** Products and coproducts are unique up to equivalence (when they exist).

**Example:** Category \( \text{Ab} \) (abelian groups)

The direct product (cartesian product) of abelian groups exists, and is an abelian group, thus it is a product in \( \text{Ab} \).

**Def:** The weak direct product of a family of groups \( \{ G_i \mid i \in I \} \), denoted \( \prod^W \{ G_i \mid i \in I \} \), is the set of \( (X_i) \) such that \( X_i = e_i \) (the identity) in \( G_i \) for all but a finite number of \( i \in I \). If all \( G_i \)'s are abelian, we write this as \( \bigoplus_{i \in I} G_i \) and call it the direct sum.
**Thm 6.5:** Let \( \{A_i : i \in I\} \) be a family of abelian groups.

Then \( \sum_{i \in I} A_i \) is a coproduct in the category of abelian groups.

**PF:** H.W.

**Note:** If \( |I| < \infty \), then the direct sum and direct product of abelian groups \( \{A_i : i \in I\} \) coincide.

**Example:** Category Grp (groups).

Again, the product of groups is simply the direct product.

**Def:** Given a family of groups \( \{G_i : i \in I\} \), let \( X = \bigcup_{i \in I} G_i \).

A word on \( X \) is any sequence \( (a_1, a_2, \ldots) \) such that

- \( a_i \in X \cap \{1\} \) and for some index \( n \), \( a_i = 1 \) if \( i \neq n \).

A word is **reduced** if

- (i) No \( a_i \in X_i \) is the identity (in \( G_i \))
- (ii) \( a_i \neq a_{i+1} \) are never in the same group \( G_j \).
- (iii) \( a_k = 1 \implies a_i = 1 \) for \( i \geq k \).

**Note:** (1, 1, 1, 1, ...) is reduced.

- We may write a reduced word uniquely as \( a_1 a_2 a_3 \ldots a_n = (a_1, a_2, a_3, \ldots, a_n, 1, 1, \ldots) \), \( a_i \in X \).

**Def:** Let \( \prod_{i \in I} G_i \) (or \( G_1 \times G_2 \times \cdots \times G_n \) if \( |I| < \infty \)) be the set of reduced words on \( X \). This is the **free product** of \( \{G_i : i \in I\} \), with binary operation concatenation (and possibly cancellation, e.g., \( a_i a_j^{-1} \)).
Example: Let \( G_1 = \text{Perm}(\{a, b, c\}) \), \( G_2 = \text{Perm}(\{1, 2, 3\}) \).

\[
g = (abc)(12)(ac)(32)(ab)(13) \in G_1 \times G_2
\]

\[
h = (13)(abc) \in G_1 \times G_2 = (bc)
\]

\[
\]

Note that \( |G_1 \times G_2| = 36 \) but \( |G_1 \times G_2| = \infty \).

Also, \( L_1 : G_1 \to G_1 \times G_2 \), \( L_1(x) = x \) (e.g., \( L_1((a5)) = a5 \)) is a monomorphism.

Def: For each \( k \in I \), \( L_k : G_k \to \prod_{i \in I} G_i \), defined by \( L_k(a) = (a, 1, 1, \ldots) \) is a monomorphism of groups.

Thm 6.6: \( \prod_{i \in I} G_i \) is a coproduct in the category of groups.

Pf: \( \text{Hw.} \)

Def: Let \( A, A_1, A_2 \) be objects in a category \( C \) and let \( f_i \in \text{Hom}_C(A, A_i) \) for \( i = 1, 2 \). A pushout (or fiber coproduct) for \( (A, A_1, A_2, f_1, f_2) \) is a commutative diagram with the following property:
Prop: Suppose $B' \xleftarrow{g_1'} A_1$ is another pushout for $(A, A_1, A_2, f_1, f_2)$. Then $B' \cong B$.

Proof: HW.

Example:

11 $\mathcal{C} = \textbf{Set}$. Let $A = A_1 \cup A_2$, $f_1 : A \rightarrow A_1$ inclusions.

Then the pushout of $(A, A_1, A_2, f_1, f_2)$ is.

Think: $A_1 \cup A_2$ with $f_1(A) \cap f_2(A_2)$ identified.
Consider 2 disjoint closed disks, $D_1$ and $D_2$. They have boundary circle $\partial D_i = S^1$. The pushout of $(S^1, D_1, D_2, i_1, i_2)$, where $i_1 : S^1 \to D_1$ are inclusion maps, is the 2-sphere.

Think: "glue 2 disks along their boundary circle."

Question: What do we get if $i_2 : S^1 \to D_2, i_2$ is the map that "wraps $S^1$ around $\partial D_2$ twice"?

13. \[ E = \text{Grp} \]

If the maps $f_{1,2}$ are injective, then the pushout is the free product with amalgamation, and

\[ P \leftarrow G_1 \]

is the quotient $G_1 *_H G_2 = (G_1 \times G_2) / N$, where $N = \langle f_1(h) f_2(h)^{-1} : h \in H \rangle$.

Note: If $H = 1$, then this is just the free product $G_1 \times G_2$. Think how this generalizes,

"The pushout without $A_i$ and $f_i : A_i \to A$ simply reduces to the coproduct of $\{A_i : i \in I\}$"
Application: Seifert–Van Kampen Theorem (algebraic topology).

Sketch of main idea: Let \( X = U \cup V \) and \( A = U \cap V \) be path-connected top spaces. The fundamental group of \( X \) is
\[
\pi_1(X, x_0) \cong \pi_1(U, x_0) \ast_{\pi_1(A, x_0)} \pi_1(V, x_0).
\]

Example: \( T^2 = \text{torus} \)

Seifert–Van Kampen. This pushout of top spaces carries over to a pushout of groups (This is a functor: \( \text{Top} \to \text{Grp} \)).

\[
\langle a, b | aba^{-1}b^{-1} \rangle \cong \mathbb{Z} \times \mathbb{Z} \quad \langle a, b | \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}
\]

\[
\pi_1(T^2, x_0) \leftarrow \pi_1(S^1 \times S^1, x_0)
\]
\[
\pi_1(D^2, x_0) \leftarrow \pi_1(A, x_0) \langle g \rangle \equiv \mathbb{Z}
\]
\[
1 \leftarrow aba^{-1}b^{-1}
\]
\[
1 \leftarrow g
\]