3. Projective and injective modules

Motivation: Free modules have some nice properties, that actually hold for a more general class of modules. Consider the following two results.

Prop 3.1: Suppose $R$ is a ring with $1$, $M$ is a unitary $R$-module, $F$ is a free $R$-module, and $f: \text{Hom}_R(M, F) \rightarrow 0$ is surjective. Then $M$ has a free submodule $E$ such that $M = E \oplus \text{ker } f$.

\[ 0 \rightarrow E \rightarrow M \xrightarrow{f} F \rightarrow 0 \]

Proof: Let $\mathcal{B}$ be a basis for $F$, choose $x_0 \in M$ s.t. $f(x_0) = 0$ for each $b \in \mathcal{B}$, and set $E = R \langle \{x_0 : b \in \mathcal{B}\} \rangle$.

If $\sum_{i=1}^{k} r_i x_i = 0$, then $0 = f(\sum_{i=1}^{k} r_i x_i) = \sum_{i=1}^{k} r_i b_i \Rightarrow r_i = 0 \forall i$, i.e., $\{x_0 : b \in \mathcal{B}\}$ is linearly independent.

Thus, $\{x_0 : b \in \mathcal{B}\}$ is a basis for $E$, so $E$ is free and clearly $E$ is $R$-isomorphic with $F$.

If $x \in M$, write $f(x) = \sum_{i=1}^{k} r_i b_i$, and note that $x - \sum_{i=1}^{k} r_i x_i \in \text{ker } f$.

Thus, $M = E + \text{ker } f$.

Since $E \cap \text{ker } f = 0$, $M = E \oplus \text{ker } f$ by Thm 2.3. \(\Box\)
Prop 3.2: Suppose $R$ is a ring with 1, and $M, N \in F$ are unitary $R$-modules with $F$ free.

If $f \in \text{Hom}_R(F, N)$ and $g \in \text{Hom}_R(M, N)$ is surjective, then $\exists h \in \text{Hom}_R(F, M)$ such that $f = gh$.

(The homomorphism $f : F \to N$ "lifts" to a homomorphism $h$.)

**Pf:** Let $B$ be a basis of $F$.

For each $b_i \in B$, choose $m_i \in M$ such that $g(m_i) = f(b_i)$.

Define $h : B \to M$ by $h(b_i) = m_i$, and extend this to $h \in \text{Hom}_R(F, M)$. This clearly works.

**Remark:** Props 3.1 & 3.2 do not necessarily hold if $F$ is not free. For example, take $M = \mathbb{Z}$, $N = F = \mathbb{Z}_2$, and $\eta : \mathbb{Z} \to \mathbb{Z}_2$ the natural quotient map, so $\text{ker } \eta = 2\mathbb{Z}$.

Then $\mathbb{Z} \not\cong \mathbb{Z} \otimes \mathbb{Z}_2$, nor does there exist $h \in \text{Hom}_\mathbb{Z}(\mathbb{Z}_2 \to \mathbb{Z})$ such that $\text{id}_{\mathbb{Z}_2} = \eta h$.

**Goal:** Understand what class of modules these results do hold for. These will be precisely the "projective" modules.

First, we need to introduce the notion of an exact sequence.
Def: A sequence of $R$-modules and $R$-homomorphisms

$$\cdots \to M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \to \cdots \quad (\star)$$

is exact at $M_i$ if $\text{im} \ f_i = \ker \ f_{i+1}$. The sequence is exact if it is exact at each $M_i$.

Prop. 3.3:

(i) $0 \to L \xrightarrow{f} M$ is exact iff $f$ is injective.

(ii) $M \xrightarrow{g} N \to 0$ is exact iff $g$ is surjective.

(iii) $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is exact iff

(a) $f$ is injective
(b) $g$ is surjective
(c) $g$ induces an isomorphism $\text{coker} \ f := M / f(L) \cong N$

PF: Exercise.

An exact sequence of type (iii) above is called a short exact sequence.

Any long exact sequence ($\star$) can be broken up into short exact sequences via restriction.

Example: If $\xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1}$ is exact, then

$$0 \to \text{im} \ f_i \to M_i \xrightarrow{\text{im} \ f_{i+1}} 0$$

is exact.
Prop 3.4: Consider a sequence \( \xymatrix{ L \ar[r]^f & M \ar[r]^g & N } \) of homomorphisms.

(i) \( f \) is injective iff \( f \circ h_1 = f \circ h_2 \Rightarrow h_1 = h_2 \) \( \forall h_i \in \text{Hom}(M, N) \).

(ii) \( g \) is surjective iff \( h_1 \circ g = h_2 \circ g \Rightarrow h_1 = h_2 \) \( \forall h_i \in \text{Hom}(L, M) \).

PF: HW #6, last semester.

Prop 3.5: Let \( \xymatrix{ 0 \ar[r] & L \ar[r]^f & M \ar[r]^g & N \ar[r] & 0 } \) be an exact sequence. Then

(i) \( \xymatrix{ 0 \ar[r] & \text{Hom}_R(D, L) \ar[r]^{f_*} & \text{Hom}_R(D, M) \ar[r]^{g_*} & \text{Hom}_R(D, N) } \) is exact.

(ii) \( \xymatrix{ 0 \ar[r] & \text{Hom}_R(N, D) \ar[r]^{g^*} & \text{Hom}_R(M, D) \ar[r]^{f^*} & \text{Hom}_R(L, D) } \) is exact.

PF:

(i) First, show exactness at \( \text{Hom}_R(D, L) \), i.e., injectivity of \( f^* \):

\[
\text{Recall } f^*: \Theta \rightarrow f \circ \Theta
\]

Suppose \( f^*(\Theta) = f^*(\Theta') \).

Then \( f \circ \Theta = f \circ \Theta' \Rightarrow \Theta = \Theta' \) since \( f \) is injective (Prop 3.4 (i)).

Next, show exactness at \( \text{Hom}_R(D, M) \), i.e., \( \text{im} f^* = \ker g^* \):

\[
\text{im} f^* \subseteq \ker g^* \quad (\text{equivalently, } g^* \circ f^* = 0);
\]

\[
f^*: \Theta \rightarrow f \circ \Theta, \quad g^*: \Theta \rightarrow g \circ \Theta
\]

\[
\Rightarrow g^* \circ f^*: \Theta \rightarrow g \circ f \circ \Theta = 0 \quad \text{since } g \circ f = 0 \quad (m f = \ker g).
\]

\( \checkmark \)
\[ \text{im } f_* \supseteq \ker g_*: \]

Suppose \( \theta \in \ker g_* \). Then \( g \circ \theta = 0 \in \text{Hom}_\#(D, N) \).

We will construct \( \varphi \in \text{Hom}_\#(D, L) \) s.t. \( f_*(\varphi) = f \circ \varphi = \theta \), i.e., a preimage of \( \theta \).

Choose \( d \in D \), and let \( m = \Theta(d) \in M \).

Since \( F \) is injective, \( \exists ! \ l \in L \) s.t. \( F(l) = m \).

Define \( \varphi(d) = F^{-1}(m) = l \).

It is easy to check that \( \varphi \in \text{Hom}_\#(D, L) \), and that \( f_*(\varphi) = F \circ \varphi = \theta \) (see above diagrams), so \( \theta \in \text{im } f_* \Rightarrow \text{im } f_* \supseteq \ker g_* \).

This proves (i). The proof of (ii) is analogous (it's dual), and is an exercise (HW).

**Def:** A short exact sequence \( 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \)

splits if \( \exists h \in \text{Hom}(N, M) \) s.t. \( g \circ h = 1_N \), i.e.,

\[ 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0, \]

This is equivalent to \( M \cong L \oplus N \), i.e., we have the following diagram.
Here $L$ is the injection of the first summand, and $\pi$ is the projection onto the second summand.

Thm 3.5: Let $P$ be a unitary $R$-module. The following are equivalent:

(i) For any unitary $R$-modules $L, M, N$, if

$$0 \longrightarrow L \overset{f}{\longrightarrow} M \overset{g}{\longrightarrow} N \longrightarrow O$$

is exact, then

$$0 \longrightarrow \text{Hom}_R(P, L) \overset{f^*}{\longrightarrow} \text{Hom}_R(P, M) \overset{g^*}{\longrightarrow} \text{Hom}_R(P, N) \longrightarrow 0$$

is exact.

(ii) Prop 3.2 holds for $P$: if $M \overset{g}{\longrightarrow} N \longrightarrow O$ is exact and $\varphi \in \text{Hom}_R(P, N)$, then $\exists h \in \text{Hom}_R(P, M)$ s.t. $\varphi = gh$ (h is a "lift" of $\varphi$).

(iii) Every short exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow P \longrightarrow 0$ splits (i.e., $M \leq L \oplus P$). Thus, Prop 3.1 holds for $P$.

(iv) $P$ is a direct summand of a free $R$-module (i.e., for some free module $F$ and $R$-module $M$, $F \cong M \oplus P$).
Pf:

(i) $\iff$ (ii) Let $\varphi \in \text{Hom}_R(P, N)$. The condition of (i) holding is equivalent to $g_1 : \text{Hom}_R(P, M) \rightarrow \text{Hom}_R(P, N)$ being surjective, i.e., there we have $g_1 : h \rightarrow \varphi$ for some $h \in \text{Hom}_R(P, M)$, such that $g_1 h = \varphi$. In other words, given the following diagram, $\exists h : P \rightarrow M$ that makes it commute. This is precisely the condition of (ii).

(ii) $\Rightarrow$ (iii) Let $0 \rightarrow L \overset{f}{\rightarrow} M \overset{g}{\rightarrow} P \rightarrow 0$ be exact. By (ii), the identity map $\text{id} : P \rightarrow P$ lifts to a homomorphism $h : P \rightarrow M$ such that $g h = \text{id}$.

Thus, we have $0 \rightarrow L \rightarrow M \overset{g}{\rightarrow} P \rightarrow 0$, as desired.

(iii) $\Rightarrow$ (iv) Every module $P$ is the quotient of a free module $F$, so we have an exact sequence $0 \rightarrow \ker \pi \rightarrow F \overset{\pi}{\rightarrow} P \rightarrow 0$ for such an $F$. Since this sequence splits by (iii), $F = P \oplus \ker \pi$.

(iv) $\Rightarrow$ (ii) Suppose that $P$ is a direct summand of a free module $F$, i.e., $F = P \oplus K$. Let $g : \text{Hom}_R(M, N)$ be surjective, and $\varphi \in \text{Hom}_R(P, N)$, we must show that there is some $h \in \text{Hom}_R(P, M)$ such that $g h = \varphi$. 

If $F$ is based on $R$, then let $\pi: P \otimes K \to P$ be the natural projection map, and $H \in \text{Hom}_R(F, M)$ such that $g \circ H = \phi \circ \pi$ (which exists by Prop 3.2).

Note that $P \cong P \otimes 0 \cong P \otimes K$, given $\pi: (p, k) \mapsto p \in P$, define $h \in \text{Hom}_R(P, M)$ by $h: (p, o) \mapsto H((p, o))$.

Clearly, this is an $R$-homomorphism and makes the above diagram commute. \( \square \)

**Def:** An $R$-module $P$ is called **projective** if it satisfies any of the equivalent conditions of Thm 3.5.

**Motivation for terminology:** $P$ is projective iff any $R$-module $M$ that projects onto $P$ has (an isomorphic copy of) $P$ as a direct summand. (Condition (iii) in Thm 3.5).

**Cor:** Every module is a quotient of a projective module.

**Examples:**

1. $\mathbb{Z}$ is a projective $\mathbb{Z}$-module;
   - Define $h(1) = g(\phi(1))$, and
   - extend additively.
$\mathbb{Z}/n\mathbb{Z}$ is not a projective $\mathbb{Z}$-module:

$$
\begin{array}{c}
0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0 \\
\end{array}
$$

Similarly, a free $\mathbb{Z}$-module cannot have nonzero elements of finite order.

The "dual" to the notion of projective module are called injective modules. Consider $f \in \text{Hom}(M, N)$.

Compare how Hom-sets induce homomorphisms between them:

$$f^*_x : \text{Hom}_R(D, M) \to \text{Hom}_R(D, N)$$

$$\varphi \longmapsto \varphi^*_x = f \circ \varphi$$

$$\begin{array}{c}
\xymatrix{
M \ar[r]^f & N \\
D \ar[ur]^{\varphi} \ar[u]_x \ar[ur]^{\varphi^*_x = f \circ \varphi} & \\
}
\end{array}$$

**Thm 3.6:** Let $Q$ be a unitary $R$-module. The following are equivalent:

1. For any unitary $R$-modules $L, M, N$, if

   $$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

   is exact, then

   $$0 \to \text{Hom}_R(N, Q) \xrightarrow{g^*_x} \text{Hom}_R(M, Q) \xrightarrow{f^*_x} \text{Hom}_R(L, Q) \to 0$$

   is exact.
(iii) If $0 \rightarrow L \xrightarrow{f} M$ is exact and $\psi \in \text{Hom}_R(L,G)$, then
\[ \exists h \in \text{Hom}_R(M,G) \text{ s.t. } \psi = hf. \]
(h is a "lift" of $\psi$).

\[ \begin{array}{c}
0 \\
\downarrow \\
Q \\
\downarrow \\
\circ
\end{array} \xrightarrow{f} \begin{array}{c}
L \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \end{array} \begin{array}{c}
M \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \end{array} \begin{array}{c}
N \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \end{array} \rightarrow 0 \]

(iii) If $Q$ is a submodule of the $R$-module $M$, then $Q$ is a direct summand of $M$, i.e., every short exact sequence
\[ 0 \rightarrow Q \rightarrow M \rightarrow N \rightarrow 0 \]
splits.

Pf: Exercise.

Def: An $R$-module $Q$ is called injective if it satisfies any of the equivalent conditions of Thm 3.6.

Example:

1. $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module, since the exact sequence
\[ 0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0 \]
does not split.

2. $Q$ and $Q/\mathbb{Z}$ are injective (but not projective) $\mathbb{Z}$-modules.

3. Fact: No non-zero finitely generated $\mathbb{Z}$-module is injective.

Cor: Every $\mathbb{Z}$-module is the submodule of an injective $\mathbb{Z}$-module.

Thm 3.7: Every unitary $R$-module $M$ is contained in an injective $R$-module. (Exercise; see Dummit and Foote Ex. 10.5 #15-16).