

MthSc 208, Spring 2011 (Differential Equations)

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HW 6

Due Monday February 14th, 2011

- (1) A lake, with volume $V = 100 \text{ km}^3$, is fed by a river at a rate of $r \text{ km}^3/\text{yr}$. In addition, there is a factory on the lake that introduces a pollutant into the lake at the rate of $p \text{ km}^3/\text{yr}$. There is another river that is fed by the lake at a rate that keeps the volume of the lake constant. This means that the rate of flow from the lake into the outlet river is $(p+r) \text{ km}^3/\text{yr}$. Let $x(t)$ denote the volume of the pollutant in the lake at time t . Then $c(t) = x(t)/V$ is the concentration of the pollutant.

- (a) Show that, under the assumption of immediate and perfect mixing of the pollutant into the lake water, the concentration satisfies the differential equation

$$c' + \frac{p+r}{V} c = \frac{p}{V}.$$

- (b) It has been determined that a concentration of over 2% is hazardous for the fish in the lake. Suppose that $r = 50 \text{ km}^3/\text{yr}$, $p = 2 \text{ km}^3/\text{yr}$, and the initial concentration of pollutant in the lake is zero. How long will it take the lake to become hazardous to the health of the fish?
- (c) Suppose that the factory from parts (a) and (b) stops operating at time $t = 0$, and that the concentration of pollutant in the lake was 3.5% at that time. Approximately how long will it take before the concentration falls below 2% and the lake is no longer hazardous for the fish?
- (2) Consider two tanks, labeled tank A and tank B for reference. Tank A contains 100 gal of solution in which is dissolved 20 lb of salt. Tank B contains 200 gal of solution in which is dissolved 40 lb of salt. Pure water flows into tank A at a rate of 5 gal/sec. There is a drain at the bottom of tank A, and the solution leaves tank A via this drain at a rate of 5 gal/sec and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B at a rate of 2.5 gal/sec. What is the salt content in tank B at the precise moment that tank B contains 250 gal of solution.
- (3) For each of the second-order differential equations below, decide whether the equation is linear or nonlinear. If the equation is linear, state whether the equation is homogeneous or inhomogeneous.
- (a) $y'' + 3y' + 5y = 3 \cos 2t$
- (b) $t^2 y'' = 4y' - \sin t$
- (c) $t^2 y'' + (1-y)y' = \cos t$
- (d) $ty'' + (\sin t)y' = 4y - \cos 5t$
- (e) $t^2 y'' + 4yy' = 0$
- (f) $y'' + 4y' + 7y = 3e^{-t} \sin t$
- (g) $y'' + 3y' + 4 \sin y = 0$
- (h) $(1-t^2)y'' = 3y$

- (4) Find the general solution to the following 2nd order linear homogeneous ODEs.

- (a) $y'' + 5y' + 6y = 0$
- (b) $y'' + y' - 12y = 0$
- (c) $y'' + 4y' + 5y = 0$
- (d) $y'' + 2y = 0$
- (e) $y'' - 4y' + 4y = 0$
- (f) $4y'' + 12y' + 9y = 0$

- (5) In this equation, we will find all solutions to the initial value problem $y'' = \lambda y$, $y(0) = y'(0) = 0$, where λ is a constant. This equation will turn up later when we study PDEs.

- (a) First, suppose that $\lambda = 0$. That is, solve $y'' = 0$, $y(0) = y'(0) = 0$.
- (b) Next, suppose $\lambda = \omega^2 > 0$.
- (i) Solve the initial value problem $y'' = \omega^2 y$, $y(0) = y'(0) = 0$.

- (ii) Let $u_1(t) = \cosh \omega t = \frac{e^{\omega t} + e^{-\omega t}}{2}$ and $u_2(t) = \sinh \omega t = \frac{e^{\omega t} - e^{-\omega t}}{2}$. Show that $u_1(t)$ and $u_2(t)$ both solve $y'' = \omega^2 y$, and use this to write the general solution of this differential equation.
- (iii) Solve the initial value problem from part (i) again, but this time, start by using the general solution you found in part (ii) (instead of exponentials).
- (c) Finally, suppose $\lambda = -\omega^2 < 0$. That is, solve $y'' = -\omega^2 y$, $y(0) = y'(0) = 0$.
- (d) Using your results from parts (a)–(c), describe all solutions to the initial value problem $y'' = \lambda y$, $y(0) = y'(0) = 0$. What are the possible values for λ ?