MthSc 208, Fall 2011 (Differential Equations)  
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HW 15  
Due Monday April 4th, 2011

(1) For each initial value problem, sketch the forcing term, and then solve for $y(t)$. Write your solution as a piecewise function (i.e., not using the Heavysie function). Recall that the function $H_{ab}(t) = H(t-a) - H(t-b)$ is the interval function.
(a) $y'' + 4y = H_{01}(t), \ y(0) = 0, \ y'(0) = 0$
(b) $y'' + 4y = tH_{01}(t), \ y(0) = 0, \ y'(0) = 0$

(2) Define the function
$$\delta_{\epsilon}^p(t) = \frac{1}{\epsilon} \left( H_p(t) - H_{p+\epsilon}(t) \right).$$
(a) Show that the Laplace transform of $\delta_{\epsilon}^p(t)$ is given by
$$\mathcal{L} \{ \delta_{\epsilon}^p(t) \} = e^{-sp} \frac{1 - e^{-s\epsilon}}{s\epsilon}.$$  
(b) Use l'Hôpital’s rule to take the limit of the result in part (a) as $\epsilon \to 0$. How does this result agree with the fact that $\mathcal{L} \{ \delta_p(t) \} = e^{-sp}$?

(3) Use a Laplace transform to solve the following initial value problem:
$$y' = \delta_p(t), \quad y(0) = 0$$
How does your answer support what engineers like to say, that the “derivative of a unit step is a unit impulse”?

(4) Define the function
$$H_{\epsilon}^p(t) = \begin{cases} 0, & 0 \leq t < p \\ \frac{1}{\epsilon} (x - p), & p \leq t < p + \epsilon \\ 1, & t \geq p + \epsilon \end{cases}$$
(a) Sketch the graph of $H_{\epsilon}^p(t)$.
(b) Without being too precise about things, we could argue that $H_{\epsilon}^p(t) \to H_p(t)$ as $\epsilon \to 0$, where $H_p(t) = H(t - p)$. Sketch the graph of the derivative of $H_{\epsilon}^p(t)$.
(c) Compare your result in (b) with the graph of $\delta_{\epsilon}^p(t)$. Argue that $H'_{\epsilon}^p(t) = \delta_{\epsilon}^p(t)$.

(5) Solve the following initial value problems.
(a) $y'' + 4y = \delta(t), \quad y(0) = 0, \quad y'(0) = 0$
(b) $y'' - 4y' - 5y = \delta(t), \quad y(0) = 0, \quad y'(0) = 0$