**MthSc 208, Fall 2011 (Differential Equations)**
**Dr. Macauley**
**HW 19**
**Due Friday April 22nd, 2011**

(1) Consider the ODE $y'' = 4y$. We know that the general solution is $y(t) = C_1e^{2t} + C_2e^{-2t}$, i.e., $\{e^{2t}, e^{-2t}\}$ is a *basis* for the solution space. Use the fact that $e^{2t} = \cosh 2t + \sinh 2t$ and $e^{-2t} = \cosh 2t - \sinh 2t$, and that any linear combination of solutions is a solution, to find two distinct solutions involving hyperbolic sines and cosines. Write the general solution using these functions.

(2) We will solve for the function $u(x, t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi, t) = 0, \qquad u(x,0) = 5\sin x + 3\sin 2x.$$

(a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of both boundary conditions (called *Dirichlet* boundary conditions) and the initial condition.
(b) Assume that $u(x,t) = f(x)g(t)$. Find $u_t$ and $u_{xx}$. Also, determine the boundary conditions for $f(x)$ (at $x = 0$ and $x = \pi$) from the boundary conditions for $u(x, t)$.
(c) Plug $u = fg$ back into the PDE and divide both sides by $c^2fg$ (i.e., "separate variables") to get the *eigenvalue problem*. Briefly justify why this quantity must be a constant. Call this constant $\lambda$. Write down two ODEs: one for $g(t)$ and one for $f(x)$.
(d) Solve for $g(t)$, $f(x)$, and $\lambda$.
(e) Using your solution to Part (d) and the principle of superposition, find the general solution to the boundary value problem.
(f) Solve the *initial value problem*, i.e., find the particular solution $u(x,t)$ that additionally satisfies $u(x, 0) = 5 \sin x + 3 \sin 2x$.
(g) What is the steady-state solution, i.e., $\lim_{t\to\infty} u(x, t)$?

(3) Consider a similar situation as the previous problem, but with slightly different boundary and initial conditions.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = 30, \quad u(\pi,t) = 100$$
$$u(x,0) = 30 + \tfrac{70}{\pi}x + 5\sin x + 3\sin 2x.$$

(a) Describe (and sketch) a physical situation that this models. Be sure to describe the impact of *both* boundary conditions and the initial condition.
(b) Use your physical intuition to determine what the steady-state solution $u_{ss}(x)$ is.
(c) Write down the solution to this initial/boundary value problem by adding the steady-state solution to the solution of the related homogeneous problem (see Part (f) of the previous problem).
(d) How does this compare to the structure of the solution to the ODE for Newton's law of heating / cooling? [*Hint*: Consider an example, e.g., $T(t) = 72 + T_h(t) = 72 + Ce^{-kt}$. Note that the heat equation is the 1-dimensional analog of Newtons law of heating / cooling (which is typically applied to a point-mass, or a "0-dimensional" object).]

(4) Consider the following initial/boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = 0, \quad u_x(\pi,t) = 0, \qquad u(x,0) = 3\sin\frac{5x}{2}.$$

(a) Describe (and sketch) a physical situation that this models. Be sure to describe the impact of *both* boundary conditions and the initial condition.

(b) Assume that there is a solution of the form $u(x,t) = f(x)g(t)$. Find $u_t$, $u_x$, and $u_{xx}$. Also, determine the boundary conditions for $f(x)$ (at $x = 0$ and $x = \pi$) from the *mixed boundary conditions* for $u(x,t)$.

(c) Plug $u = fg$ back into the PDE and divide both sides by $c^2 fg$ (i.e., "separate variables") to get the eigenvalue problem. Write down two ODEs: one for $g(t)$ and one for $f(x)$.

(d) Solve the ODEs from the previous part for $f$ and $g$. You may assume that $\lambda = -\omega^2$, (i.e., that $\lambda < 0$). Determine $\omega$ (be sure to show your work for this part, the answer may surprise you!).

(e) Write down the general solution $u(x,t)$ for the boundary value problem.

(f) Find the particular solution for $u(x,t)$ that additionally satisfies the initial condition $u(x,0) = 3\sin(5x/2)$.

(g) What is the steady-state solution?

(5) Let $u(x,t)$ be the temperature of a bar of length 10, at position $x$ and time $t$ (in hours). Suppose that initially, the temperature increases linearly from 70° at the left endpoint to 80° at the other end. Furthermore, suppose that the temperature at the left end of the bar is held at a constant 70 degrees, and that the right end is insulated so no heat can escape. Finally, suppose that the interior of the bar is poorly insulated, so heat escapes from it, causing the temperature to decrease at an additional constant rate of 1° per hour. Write an initial/boundary value problem for $u(x,t)$ that could model this situation.