MthSc 208, Fall 2011 (Differential Equations)
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HW 20
Due Monday April 25th, 2011

(1) Let $u(x,t)$ be the temperature of a bar of length 10, that is insulated so that no heat can enter or leave. Suppose that initially, the temperature increases linearly from $70°$ at one endpoint, to $80°$ at the other endpoint.
  (a) Sketch the initial heat distribution on the bar, and express it as a function of x.
  (b) Write down an initial/boundary value problem to which $u(x,t)$ is a solution (Let the constant from the heat equation be c^2).
  (c) What will the steady-state solution be?

(2) Consider the following PDE:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t)=0, \quad u_x(\pi,t)+\gamma\, u(\pi,t)=0, \qquad u(x,0)=h(x)\ ,$$

where γ is a non-negative constant, and $h(x)$ and arbitrary function on $[0,\pi]$
  (a) Describe a physical situation that this models. Be sure to describe the impact of the initial condition, *both* boundary conditions and the constant γ.
  (b) What is the steady-state solution, and why? (Use your physical intuition).

(3) We will solve the heat equation with *periodic boundary conditions*:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial \theta^2}, \qquad u(\theta+2\pi,t)=u(\theta,t), \qquad u(\theta,0)=2+4\sin 3\theta-\cos 5\theta.$$

  (a) Describe and sketch a situation that this models.
  (b) Assume that there is a solution of the form $u(\theta,t)=f(\theta)g(t)$. Find u_t and $u_{\theta\theta}$. Use the periodic boundary conditions for $u(\theta,t)$ to derive similar periodic boundary conditions for $f(\theta)$.
  (c) Plug $u=fg$ back into the PDE and divide both sides by c^2fg (i.e., "separate variables") to get the eigenvalue problem. Write down two ODEs: one for $g(t)$ and one for $f(\theta)$.
  (d) Solve for $g(t)$, $f(\theta)$, and λ. *Note: You won't be able to conclude that $a=0$ or $b=0$ – so unlike before, they'll both stick around.*
  (e) Find the general solution of the boundary value problem. As before, it will be a superposition (infinite sum) of solutions $u_n(\theta,t)=f_n(\theta)g_n(t)$.
  (f) Find the particular solution to the initial value problem that satisfies the initial condition $u(\theta,0)=2+4\sin 3\theta-\cos 5\theta$.
  (g) What is the steady-state solution? Give a mathematical *and* intuitive (physical) justification for this.

(4) Find the function $u(x,t)$ defined for $0\le x\le\pi$ and $t\ge 0$ which satisfies the following initial value problem of the *wave equation*:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t)=u(\pi,t)=0,$$
$$u(x,0)=8\sin x+11\sin 2x+15\sin 4x, \qquad u_t(x,0)=0.$$

  (a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of both boundary conditions and both initial conditions.
  (b) Assume that $u(x,t)=f(x)g(t)$. Find u_t, u_{tt}, and u_{xx}. Also, determine two boundary conditions for $f(x)$ (at $x=0$ and $x=\pi$) from the boundary conditions for $u(x,t)$, and one boundary condition for $g(t)$.

(c) Plug $u = fg$ back into the PDE and divide both sides by $c^2 fg$ (i.e., "separate variables") to get the eigenvalue problem. Write down two ODEs: one for $g(t)$ and one for $f(x)$.
(d) Write down the solution of the ODE for $f(x)$, and λ (this is the same as in the heat equation; there is no need to re-derive it). Solve the ODE for $g(t)$.
(e) Using your solution to (d) and the principle of superposition, find the general solution to the initial/boundary value problem. As before, it will be a superposition (infinite sum) of solutions $u_n(x,t) = f_n(x)g_n(t)$.
(f) Solve the *initial value problem*, i.e., find the particular solution $u(x,t)$ that additionally satisfies $u(x,0) = 8\sin x + 11\sin 2x + 15\sin 4x$.
(g) What is the long-term behavior of this solution (i.e., what happens as $t \to \infty$)?

(5) In this problem, we will find the function $u(x,t)$ defined for $0 \leq x \leq \pi$ and $t \geq 0$ which satisfies the following conditions:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = x(\pi - x), \qquad u_t(x,0) = 0.$$

Steps (a)–(e) are the same as in the previous problem, and need not be repeated. Instead, repeat part (f) with these new initial conditions. Sketch this scenerio at time $t = 0$.

(6) In this problem, we will find the function $u(x,t)$ defined for $0 \leq x \leq \pi$ and $t \geq 0$ which satisfies different initial conditions:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = 0, \qquad u_t(x,0) = x(\pi - x).$$

Steps (a)–(c) are the same as in previous problems, and need not be repeated. Instead, repeat parts (d), (e), and (f) with these new initial conditions. What physical situation does this model? Give a physical interpretation for both boundary conditions, and both initial conditions, and sketch this scenerio at time $t = 0$.