MthSc 208, Fall 2011 (Differential Equations)
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HW 21
Due Friday April 29th, 2011

(1) Which of the following functions are harmonic?
  (a) $f(x) = 10 - 3x$.
  (b) $f(x, y) = x^2 + y^2$.
  (c) $f(x, y) = x^2 - y^2$.
  (d) $f(x, y) = e^x \cos y$.
  (e) $f(x, y) = x^3 - 3xy^2$.

(2) (a) Solve the following Dirichlet problem for Laplace's equation in a square region: Find $u(x, y)$, $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0, y) = u(\pi, y) = 0,$$
$$u(x, 0) = 0, \quad u(x, \pi) = 4 \sin x - 3 \sin 2x + 2 \sin 3x.$$

  (b) Solve the following Dirichlet problem for Laplace's equation in the same square region: Find $u(x, y)$, $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0, y) = 0, \quad u(\pi, y) = y(\pi - y),$$
$$u(x, 0) = u(x, \pi) = 0$$

  (c) By adding the solutions to parts (a) and (b) together (superposition), find the solution to the Dirichlet problem: Find $u(x, y)$, $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0, y) = 0, \quad u(\pi, y) = y(\pi - y),$$
$$u(x, 0) = 0, \quad u(x, \pi) = 4 \sin x - 3 \sin 2x + 2 \sin 3x.$$

  (d) Sketch the solutions to (a), (b), and (c). *Hint: it is enough to sketch the boundaries, and then use the fact that the solutions are harmonic functions.*
  (e) Consider the heat equation in a square region, along with the following boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad u(0, y) = 0, \quad u(\pi, y) = y(\pi - y),$$
$$u(x, 0) = 0, \quad u(x, \pi) = 4 \sin x - 3 \sin 2x + 2 \sin 3x.$$

  What is the steady-state solution? (Note: This will *not* depend on the initial conditions!)

(3) Consider the following initial/boundary value problem for the heat equation in a square region, and the funtion $u(x, y, t)$, where $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ and $t \geq 0$.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$
$$u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0$$
$$u(x, y, 0) = 2 \sin x \sin y + 5 \sin 2x \sin y.$$

  (a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of the boundary conditions and the initial condition.
  (b) Assume that the solution has the form $u(x, y, t) = f(x, y)g(t)$. Find u_{xx}, u_{yy}, and u_t.

(c) Plug $u = fg$ back into the PDE and divide both sides by fg (i.e., "separate variables") to get the *eigenvalue problem*. Briefly justify why this quantity must be a constant. Call this constant λ. Write down an ODE for $g(t)$, and a PDE for $f(x, y)$ (the *Helmholz equation*). Include four boundary conditions for $f(x, y)$.
(d) Solve the Helmholz equation and determine λ. You may assume that $f(x, y) = X(x)Y(y)$.
(e) Solve the ODE for $g(t)$.
(f) Find the general solution of the boundary value problem. It will be a superposition (infinite sum) of solutions $u_{nm}(x, y, t) = f_{nm}(x, y)g_{nm}(t)$.
(g) Find the particular solution to the initial value problem that additionally satisfies the initial condition $u(x, y, 0) = 2\sin x \sin y + 5\sin 2x \sin y$.
(h) What is the steady-state solution? Give a mathematical *and* intuitive (physical) justification.

(4) Consider the following initial/boundary value problem for the heat equation in a square region, and the funtion $u(x, y, t)$, where $0 \le x \le \pi$, $0 \le y \le \pi$ and $t \ge 0$.

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
&u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0 \\
&u(x, y, 0) = (7\sin x)\, y(\pi - y).
\end{aligned}$$

Since the only difference between this problem and the previous one is in the initial condition, steps (b)–(f) are the same and need not be repeated. Briefly describe, and sketch, a physical situation which this models, and then carry out steps (g) and (h), given this new initial condition.

(5) Consider the 2D heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

subject to the boundary conditions

$$\begin{aligned}
&u(x, 0, t) = u(0, y, t) = u(\pi, y, t) = 0 \\
&u(x, \pi, t) = x(\pi - x).
\end{aligned}$$

(a) What is the steady-state solution, $u_{ss}(x, y)$? [Hint: Look at a previous problem on Laplace's equation]. Sketch it.
(b) Write down the general solution this this boundary value problem by adding $u_{ss}(x, y)$ to the general solution of a related *homogeneous* boundary value problem [Hint: Look at a previous problems on the 2D heat equation].

(6) Solve the following initial value problem for a vibrating square membrane: Find $u(x, y, t)$, $0 \le x \le \pi$, $0 \le y \le \pi$ such that

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \\
&u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0 \\
&u(x, y, 0) = p(x)q(y), \qquad u_t(x, y, 0) = 0.
\end{aligned}$$

where

$$p(x) = \begin{cases} x, & \text{for } 0 \le x \le \pi/2, \\ \pi - x, & \text{for } \pi/2 \le x \le \pi, \end{cases} \quad , \qquad q(y) = \begin{cases} y, & \text{for } 0 \le y \le \pi/2, \\ \pi - y, & \text{for } \pi/2 \le y \le \pi. \end{cases}$$

(a) Briefly describe a physical situation which this models. Be sure to explain the effect of the boundary conditions and the initial condition. Sketch the initial displacement, $u(x, y, 0)$.

(b) Assume that the solution has the form $u(x, y, t) = f(x, y)g(t)$. Find u_{xx}, u_{yy}, u_t, and u_{tt}.

(c) Plug $u = fg$ back into the PDE and divide both sides by fg (i.e., "separate variables") to get the *eigenvalue problem*. Briefly justify why this quantity must be a constant. Call this constant λ. Write down an ODE for $g(t)$, and a PDE for $f(x, y)$ (the *Helmholz equation*). Include four boundary conditions for $f(x, y)$ and one for $g(t)$.

(d) You may assume that $\lambda = -(n^2+m^2)$, and that the solution to the Helmholz equation is $f(x, y) = b_{nm} \sin nx \sin my$. Solve the ODE for $g(t)$, using the initial condition.

(e) Find the general solution of the boundary value problem. It will be a superposition (infinite sum) of solutions $u_{nm}(x, y, t) = f_{nm}(x, y)g_{nm}(t)$.

(f) Find the particular solution to the initial value problem that additionally satisfies the initial condition $u(x, y, 0) = p(x)q(y)$.

(g) What is the long-term behavior of $u(x, y, t)$, i.e., as $t \to \infty$. Give a mathematical *and* intuitive (physical) justification.