

MthSc 208: Differential Equations (Spring 2011)

In-class Worksheet 7b: The Wave Equation

NAME:

We will solve for the function $u(x, t)$ defined for $0 \leq x \leq \pi$ and $t \geq 0$ which satisfies the following initial value problem of the wave equation:

$$u_{tt} = c^2 u_{xx} \quad u(0, t) = u(\pi, t) = 0, \quad u(x, 0) = x(\pi - x), \quad u_t(x, 0) = 1.$$

- (a) Carefully describe (and sketch) a physical situation that this models.
- (b) Assume that $u(x, t) = f(x)g(t)$. Compute u_t , u_{tt} , u_x , u_{xx} , and find boundary conditions for $f(x)$.
- (c) Plug $u = fg$ back into the PDE and separate variables by dividing both sides of the equation by $c^2 fg$. Set this equal to a constant λ , and write down two ODEs: one for $f(x)$ and one for $g(t)$.

(d) Solve the ODE for $f(x)$ (including the boundary conditions), and determine λ . You may assume that $\lambda = -\omega^2 < 0$.

(e) Now that you know what λ is, solve the ODE for $g(t)$.

(f) Find the general solution of the PDE. As before, it will be a superposition (infinite sum) of solutions $u_n(x, t) = f_n(x)g_n(t)$.

- (g) Find the particular solution to the initial value problem by using the initial conditions. The following information is useful:

The Fourier sine series of $x(\pi - x)$ is $\sum_{n=1}^{\infty} \frac{4}{\pi n^3} (1 - (-1)^n) \sin nx$.

- (h) What is the long-term behavior of the system? Give a mathematical, and physical, justification.