1. Let S be a set of vectors in a finite-dimensional vector space X. Show that S is a basis of X if every vector of X can be written in one and only one way as a linear combination of the vectors in S.

2. Let X be a finite-dimensional vector space over K and let $\{x_1, \ldots, x_n\}$ be an ordered basis for X. Let U be a vector space over the same field K but possibly with a different dimension, and let $\{u_1, \ldots, u_n\}$ be an arbitrary set of vectors in U. Show that there is precisely one linear transformation $T\colon X \to U$ such that $Tx_i = u_i$ for each $i = 1, \ldots, n$.

3. Let K be a finite field. The *characteristic* of K, denoted $\operatorname{char} K$, is the smallest positive integer n for which $\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0$.

   (a) Prove that the characteristic of K is prime.

   (b) Show that K is a vector space over $\mathbb{Z}_p$, where $p = \operatorname{char} K$.

   (c) Show that the order $|K|$ of K (the number of elements it contains) is a prime power.

   (d) Show that if K and L are finite fields with $K \subset L$ and $|K| = p^m$ and $|L| = p^n$, then m divides n.

4. Let S be a subset of X. Recall that the *annihilator* of S is the set $S^\perp = \{\ell \in X' \mid \ell(s) = 0 \text{ for all } s \in S\}$.

   (a) Show that if S is a subspace of X, then $S^\perp$ is a subspace of X'.

   (b) Let Y be the smallest subspace of X that contains S. Show that $S^\perp = Y^\perp$.

5. Let X be a vector space over a field K.

   (a) Let $v_1, \ldots, v_n$ be a basis for X. For each i, show that there exists a unique linear map $f_i\colon X \to K$ such that $f_i(v_i) = 1$ and $f_i(v_j) = 0$ for $j \neq i$.

   (b) Show that $f_1, \ldots, f_n$ is a basis for X' (called the *dual basis* of $v_1, \ldots, v_n$).

   (c) Consisder the basis $v_1 = (1, -1, 3)$, $v_2 = (0, 1, -1)$, and $v_3 = (0, 3, -2)$ of $X = \mathbb{R}^3$. Find a formula for each element of the dual basis.

   (d) Express the linear map $f \in X'$, where $f(x, y, z) = 2x - y + 3z$ as a linear combination of the dual basis, f_1, f_2, f_3.

6. Let $\mathcal{P}_2$ be the vector space of all polynomials $p(x) = a_0 + a_1x + a_2x^2$ over $\mathbb{R}$, with degree ≤ 2. Let ξ_1, ξ_2, ξ_3 be distinct real numbers, and define
$$\ell_j = p(\xi_j) \quad \text{for} \quad j = 1, 2, 3\,.$$

   (a) Show that ℓ_1, ℓ_2, ℓ_3 are linearly independent functions on $\mathcal{P}_2$.

   (b) Show that ℓ_1, ℓ_2, ℓ_3 is a basis for the dual space $\mathcal{P}_2'$.

   (c) Find polynomials $p_1(x), p_2(x), p_3(x)$ in $\mathcal{P}_2$ of which ℓ_1, ℓ_2, ℓ_3 is the dual basis in $\mathcal{P}_2'$.

7. Let W be the subspace of $\mathbb{R}^4$ spanned by $(1, 0, -1, 2)$ and $(2, 3, 1, 1)$. Which linear functions $\ell(x) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ are in the annihilator of W?