1. Let $T: X \to U$ be a linear map. Prove the following:

(a) The image of a subspace of $X$ is a subspace of $U$.
(b) The inverse image of a subspace of $U$ is a subspace of $X$.

2. Prove Theorem 3.3 in Lax:

(a) The composite of linear mappings is also a linear mapping.
(b) Composition is distributive with respect to the addition of linear maps, that is,

\[(R + S) \circ T = R \circ T + S \circ T\]

and

\[S \circ (T + P) = S \circ T + S \circ P,\]

where $R, S: U \to V$ and $P, T: X \to U$.

3. Suppose that $X$, $U$, and $V$ are vector spaces, and $S: X \to U$ and $T: X \to V$ are linear maps, and $T$ is surjective. Give necessary and sufficient conditions for the existence of a well-defined linear map $R: U \to V$ such that $RS = T$. Prove all of your claims.

4. Let $X$ be a finite-dimensional vector space over $K$ and let $\{x_1, \ldots, x_n\}$ be an ordered basis for $X$. Let $U$ be a vector space over the same field $K$ but possibly with a different dimension, and let $\{u_1, \ldots, u_n\}$ be an arbitrary set of vectors in $U$. Show that there is precisely one linear transformation $T: X \to U$ such that $Tx_i = u_i$ for each $i = 1, \ldots, n$. 