1. Show that whenever meaningful,
   
   (i) \((ST)′ = T′S′\)
   
   (ii) \((T + R)′ = T′ + R′\)
   
   (iii) \((T^{-1})′ = (T′)^{-1}\).

   Here, \(S′\) denotes the transpose of \(S\). Carefully describe what you mean by “whenever meaningful” in each case.

2. Give a direct algebraic proof of \(N_{T'} = R_T\) (i.e., don’t just use the fact that \(N_{T'} = R_T\) and take the annihilator of both sides.)

3. Let \(X\) be a finite-dimensional vector space and \(A, B \in \mathcal{L}(X, X)\) (linear functions \(X \to X\)).

   (a) Show that if \(A\) is invertible and similar to \(B\), then \(B\) is also invertible, and \(B^{-1}\) is similar to \(A^{-1}\).

   (b) Show that if either \(A\) or \(B\) is invertible, then \(AB\) and \(BA\) are similar.

4. Suppose \(T: X \to X\) is a linear map of rank 1, and \(\dim X < \infty\).

   (a) Show that there exists a unique \(c \in K\) such that \(T^2 = cT\).

   (b) Show that if \(c \neq 1\), then \(I - T\) has an inverse.

5. Suppose that \(S, T: X \to X\) and \(\dim X < \infty\).

   (a) Show that \(\text{rank}(S + T) \leq \text{rank}(S) + \text{rank}(T)\).

   (a) Show that \(\text{rank}(ST) \leq \text{rank}(S)\).

   (b) Show that \(\dim(N_{ST}) \leq \dim N_S + \dim N_T\).