1. Let $T\colon X \to U$, with $\dim X = n$ and $\dim U = m$. Show that there exist bases B for X and B' for U such that the matrix of T in block form is

$$M = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix.

2. Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ with matrix representation $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ -3 & 0 & 3 \end{pmatrix}$ with respect to the standard basis. What is the matrix representation of T with respect to the basis $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$?

3. Let $\mathcal{P}_n = \{p(x) \in \mathbb{R}[x] \mid \deg(p(x)) < n\}$.

   (a) Show that the map $T\colon \mathcal{P}_3 \to \mathcal{P}_4$ given by

   $$T(p(x)) = 6 \int_1^x p(t)\, dt$$

   is linear. Indicate whether it is 1–1 or onto.

   (b) Let $B_3 = \{1, x, x^2\}$ be a basis for $\mathcal{P}_3$ and let $B_4 = \{1, x, x^2, x^3\}$ be a basis for $\mathcal{P}_4$. Find the matrix representation of T with respect to these bases.

4. Let T be a matrix over a field K.

   (a) Prove that if T has a left inverse, then $Tx = u$ has at most one solution.

   (b) Prove that if T has a right inverse, then $Tx = u$ has at least one solution.

   (c) What are the possibilities for the rank of T if it has a left inverse? What if it has a right inverse?

5. Let A be an $m \times n$ matrix, and $b \in \mathbb{R}^n$.

   (a) Prove that $Ax = b$ is solvable if and only if b is in the column space of A.

   (b) Identify geometrically, as clearly as you can, the subset of 3-dimensional Euclidean space of $\mathbb{R}^3$ that corresponds to the column space of the matrix

   $$A = \begin{pmatrix} 0 & 2 & -2 \\ 3 & 5 & -2 \\ 4 & 0 & 4 \end{pmatrix}.$$

6. Find necessary and sufficient conditions on the entries u_1, u_2, u_3, u_4 under which the following system of linear equations will have at least one solution over $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, and give the number of solutions in case the conditions are met.

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 3 \\ 1 & -1 & -1 & 2 \\ 0 & 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

7. Consider the system of linear equations $Tx = u$, where

$$T = \begin{pmatrix} 0 & 2 & -2 \\ 3 & 5 & -2 \\ 4 & 0 & 4 \end{pmatrix}, \qquad u = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

   (a) Solve $Tx = u$ over $\mathbb{Q}$.

   (b) Solve $Tx = u$ over $\mathbb{Z}_2$.