1. Let $T : X \to U$, with $\dim X = n$ and $\dim U = m$. Show that there exist bases $B$ for $X$ and $B'$ for $U$ such that the matrix of $T$ in block form is

$$M = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix},$$

where $I_k$ is the $k \times k$ identity matrix.

2. Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ with matrix representation

$$
\begin{pmatrix}
1 & -1 & 0 \\
0 & 2 & -2 \\
-3 & 0 & 3
\end{pmatrix}
$$

with respect to the standard basis. What is the matrix representation of $T$ with respect to the basis $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$?

3. Let $\mathcal{P}_n = \{ p(x) \in \mathbb{R}[x] \mid \deg(p(x)) < n \}$.

(a) Show that the map $T : \mathcal{P}_3 \to \mathcal{P}_4$ given by

$$T(p(x)) = 6 \int_1^x p(t) \, dt$$

is linear. Indicate whether it is 1–1 or onto.

(b) Let $B_3 = \{1, x, x^2\}$ be a basis for $\mathcal{P}_3$ and let $B_4 = \{1, x, x^2, x^3\}$ be a basis for $\mathcal{P}_4$. Find the matrix representation of $T$ with respect to these bases.

4. Let $T$ be a matrix over a field $K$.

(a) Prove that if $T$ has a left inverse, then $Tx = u$ has at most one solution.

(b) Prove that if $T$ has a right inverse, then $Tx = u$ has at least one solution.

(c) What are the possibilities for the rank of $T$ if it has a left inverse? What if it has a right inverse?

5. Let $A$ be an $m \times n$ matrix, and $b \in \mathbb{R}^n$.

(a) Prove that $Ax = b$ is solvable if and only if $b$ is in the column space of $A$.

(b) Identify geometrically, as clearly as you can, the subset of 3-dimensional Euclidean space of $\mathbb{R}^3$ that corresponds to the column space of the matrix

$$A = \begin{pmatrix}
0 & 2 & -2 \\
3 & 5 & -2 \\
4 & 0 & 4
\end{pmatrix}.$$
6. Find necessary and sufficient conditions on the entries $u_1, u_2, u_3, u_4$ under which the following system of linear equations will have at least one solution over $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, and give the number of solutions in case the conditions are met.

\[
\begin{pmatrix}
1 & 2 & 1 & 1 \\
2 & 1 & 0 & 3 \\
1 & -1 & -1 & 2 \\
0 & 3 & 2 & -1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
= 
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{pmatrix}
\]

7. Consider the system of linear equations $Tx = u$, where

\[
T = \begin{pmatrix}
0 & 2 & -2 \\
3 & 5 & -2 \\
4 & 0 & 4
\end{pmatrix}, \quad u = \begin{pmatrix}
1 \\
-1 \\
1
\end{pmatrix}.
\]

(a) Solve $Tx = u$ over $\mathbb{Q}$.
(b) Solve $Tx = u$ over $\mathbb{Z}_2$. 
