1. Prove the following properties of the trace function:
   (a) $\operatorname{tr} AB = \operatorname{tr} BA$ for all $m \times n$ matrices A and $n \times m$ matrices B.
   (b) $\operatorname{tr} AA^T = \sum a_{ij}^2$ for all $n \times n$ matrices A.

2. Find the eigenvalues and corresponding eigenvectors for the following matrices over $\mathbb{C}$.

$$(a)\quad \begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix} \qquad (b)\quad \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} \qquad (c)\quad \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix} \qquad (d)\quad \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}.$$

3. (a) Show that if A and B are similar, then A and B have the same eigenvalues.
   (b) Is the converse of Part (a) true? Prove or disprove.

4. Let A_ϕ be the 3×3 matrix representing a rotation of $\mathbb{R}^3$ through an angle ϕ about the y-axis.
   (a) Find the eigenvalues for A_ϕ over $\mathbb{C}$.
   (b) Determine necessary and sufficient conditions on ϕ in order for A_ϕ to have 3 linearly independent eigenvectors in $\mathbb{R}^3$. Justify your claim and interpret it geometrically.

5. Let A be a 2×2 matrix over $\mathbb{R}$ satisfying $A^T = A$. Prove that A has 2 linearly independent eigenvectors in $\mathbb{R}^2$.

6. Let A be an $n \times n$ matrix over $\mathbb{C}$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. For a vector $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, define the *norm* of z by

$$||z|| = \left(\sum_{i=1}^{n} |z_i| \right)^{1/2}.$$

   (a) Prove that if $|\lambda_i| < 1$ for all i, then $||A^N z|| \to 0$ as $N \to \infty$ for all $z \in \mathbb{C}^n$.
   (b) Prove that if $|\lambda_i| > 1$ for all i, then $||A^N z|| \to \infty$ as $N \to \infty$ for all $z \in \mathbb{C}^n$.

7. Let A be an $n \times n$ matrix over $\mathbb{C}$ with an eigenvalue λ with index $m \geq 2$. Let v_1 be a corresponding eigenvector, and let v_2 be a generalized eigenvector such that $(A - \lambda I)v_2 = v_1$.
   (a) Prove that for any natural number N,

$$A^N v_2 = \lambda^N v_2 + N\lambda^{N-1} v_1 .$$

   (b) Prove that for any polynomial $q(t) \in \mathbb{C}[t]$,

$$q(A)v_2 = q(\lambda)v_2 + q'(\lambda)v_1 ,$$

   where $q'(t)$ is the derivative of q.

(c) Conjecture a formula for $q(A)v_m$, where $v_1, \dots, v_m$ are generalized eigenvectors of A with $(A - \lambda I)v_k = v_{k-1}$ (and say $v_0 = 0$, for convenience).

8. Let A be an invertible $n \times n$ matrix. Prove that A^{-1} can be written as a polynomial in degree at most $n-1$. That is, prove that there are scalars a_i such that

$$A^{-1} = a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots a_1 A + a_0 I \,.$$