1. Consider the following matrices:

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}.$$

   (a) Determine the characteristic and minimal polynomials of A, B, and C.

   (b) Determine the eigenvectors and generalized eigenvectors of A, B, and C.

2. Consider the following matrices:

$$A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -4 & 85 \\ 1 & 4 & -30 \\ 0 & 0 & 3 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}.$$

   A straightforward calculation shows that the characteristic polynomials are

$$p_A(t) = p_B(t) = p_C(t) = (t-2)^2(t-3).$$

   (a) Determine the minimal polynomials $m_A(t)$, $m_B(t)$, and $m_C(t)$.

   (b) Determine the eigenvectors and generalized eigenvectors of A, B, and C.

   (c) Determine which of these matrices are similar.

3. Compute the Jordan canonical form of the following matrices:

$$A = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & -6 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & -4 \\ 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. Let λ be an eigenvalue of A, and let N_i be the nullspace of $(A-\lambda I)^i$. Prove that $A-\lambda I$ extends to a well-defined map $N_{i+1}/N_i \longrightarrow N_i/N_{i-1}$, and that this mapping is 1–1.

5. Let A be a matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_k$, and denote the index of λ_i by d_i.

   (a) Prove, without using Jordan canonical form, that the minimal polynomial of A is

$$m_A(t) = \prod_{i=1}^{k} (t-\lambda_i)^{d_i}.$$

   (b) Give a simple proof using the Jordan canoncial form.

6. Find a list of real matrices, as long as possible, such that

   (i) The characteristic polynomial of each matrix is $(x-1)^5(x+1)$

   (ii) The minimal polynomial of each matrix is $(x-1)^2(x+1)$

   (iii) No two matrices in the list are similar to each other.

7. Let A be an $n \times n$ matrix over $\mathbb{C}$.
   (a) Prove that if $A^k = A$ for some integer $k > 1$, then A is diagonalizable.
   (b) Prove that if $A^k = 0$, then $A^n = 0$.

8. Let $X \subset \mathbb{R}[x, y]$ be the space of polynomials in x, y of total degree $\leq n$. Show that the map
$$A : X \longrightarrow X, \qquad f \longmapsto f + \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$
is linear, and find its Jordan canonical form.

9. Let X be an n-dimensional vector space over $\mathbb{C}$, and let $A, B \colon X \to X$ be linear maps.
   (a) Prove that if $AB = BA$, then for any eigenvector v of A with eigenvalue λ, the vector Bv is an eigenvector of A for λ.
   (b) Show that if $\{A_1, A_2, \dots \mid A_i \colon X \to X\}$ is a (possibly infinite) set of pairwise commuting maps, then there is a nonzero $x \in X$ that is an eigenvector of every A_i.
   (c) Suppose that A and B are both diagonalizable. Show that $AB = BA$ if and only if they are *simultaneously diagonalizable*, i.e., there exists an invertible $n \times n$-matrix P such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.

10. Let X be an n-dimensional vector space, and $A \colon X \to X$ a linear map with distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Let $v_1, \dots, v_n$ be the corresponding eigenvectors of A, and let $\ell_1, \dots, \ell_n$ be the corresponding eigenvectors of the transpose $A' \colon X' \to X'$.
    (a) Prove that $(\ell_i, v_i) \neq 0$ for $i = 1, \dots, n$.
    (b) Show that if $x = a_1 v_1 + \cdots a_n v_n$, then $a_i = (\ell_i, x)/(\ell_i, v_i)$.
    (c) Is $\ell_1, \dots, \ell_n$ necessarily the dual basis of $v_1, \dots, v_n$? Why or why not?