1. Consider the following matrices:

\[ A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}. \]

(a) Determine the characteristic and minimal polynomials of \( A \), \( B \), and \( C \).

(b) Determine the eigenvectors and generalized eigenvectors of \( A \), \( B \), and \( C \).

2. Consider the following matrices:

\[ A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -4 & 85 \\ 1 & 4 & -30 \\ 0 & 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}. \]

A straightforward calculation shows that the characteristic polynomials are

\[ p_A(t) = p_B(t) = p_C(t) = (t - 2)^2(t - 3). \]

(a) Determine the minimal polynomials \( m_A(t) \), \( m_B(t) \), and \( m_C(t) \).

(b) Determine the eigenvectors and generalized eigenvectors of \( A \), \( B \), and \( C \).

(c) Determine which of these matrices are similar.

3. Compute the Jordan canonical form of the following matrices:

\[ A = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & -6 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & -4 \\ 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

4. Let \( \lambda \) be an eigenvalue of \( A \), and let \( N_i \) be the nullspace of \( (A - \lambda I)^i \). Prove that \( A - \lambda I \) extends to a well-defined map \( N_{i+1}/N_i \to N_i/N_{i-1} \), and that this mapping is 1–1.

5. Let \( A \) be a matrix with distinct eigenvalues \( \lambda_1, \ldots, \lambda_k \), and denote the index of \( \lambda_i \) by \( d_i \).

(a) Prove, without using Jordan canonical form, that the minimal polynomial of \( A \) is

\[ m_A(t) = \prod_{i=1}^{k} (t - \lambda_i)^{d_i}. \]

(b) Give a simple proof using the Jordan canonical form.

6. Find a list of real matrices, as long as possible, such that

(i) The characteristic polynomial of each matrix is \( (x - 1)^5(x + 1) \)

(ii) The minimal polynomial of each matrix is \( (x - 1)^2(x + 1) \)

(iii) No two matrices in the list are similar to each other.
7. Let $A$ be an $n \times n$ matrix over $\mathbb{C}$.
   (a) Prove that if $A^k = A$ for some integer $k > 1$, then $A$ is diagonalizable.
   (b) Prove that if $A^k = 0$, then $A^n = 0$.

8. Let $X \subset \mathbb{R}[x,y]$ be the space of polynomials in $x, y$ of total degree $\leq n$. Show that the map
   \[ A : X \longrightarrow X, \quad f \mapsto f + \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \]
   is linear, and find its Jordan canonical form.

9. Let $X$ be an $n$-dimensional vector space over $\mathbb{C}$, and let $A, B : X \rightarrow X$ be linear maps.
   (a) Prove that if $AB = BA$, then for any eigenvector $v$ of $A$ with eigenvalue $\lambda$, the vector $Bv$ is an eigenvector of $A$ for $\lambda$.
   (b) Show that if $\{A_1, A_2, \ldots \mid A_i : X \rightarrow X\}$ is a (possibly infinite) set of pairwise commuting maps, then there is a nonzero $x \in X$ that is an eigenvector of every $A_i$.
   (c) Suppose that $A$ and $B$ are both diagonalizable. Show that $AB = BA$ if and only if they are simultaneously diagonalizable, i.e., there exists an invertible $n \times n$-matrix $P$ such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.

10. Let $X$ be an $n$-dimensional vector space, and $A : X \rightarrow X$ a linear map with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $v_1, \ldots, v_n$ be the corresponding eigenvectors of $A$, and let $\ell_1, \ldots, \ell_n$ be the corresponding eigenvectors of the transpose $A' : X' \rightarrow X'$.
   (a) Prove that $(\ell_i, v_i) \neq 0$ for $i = 1, \ldots, n$.
   (b) Show that if $x = a_1 v_1 + \cdots a_n v_n$, then $a_i = (\ell_i, x)/(\ell_i, v_i)$.
   (c) Is $\ell_1, \ldots, \ell_n$ necessarily the dual basis of $v_1, \ldots, v_n$? Why or why not?