1. Consider the following matrix:

\[ M_n = \begin{pmatrix} 0 & -a_0 \\ I_{n-1} & -a_{n-1} \end{pmatrix} \]

where \( a_n = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \).

(a) Show that the characteristic polynomial of \( M_n \) is

\[ P_{M_n}(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0. \]

Here, \( I_{n-1} \) denotes the \((n-1) \times (n-1)\) identity matrix.

(b) Is \( P_{M_n}(t) \) also the minimal polynomial? Prove or disprove.

(c) Now, let \( X \) be a 4 dimensional vector space over \( \mathbb{R} \) with basis \( \{ x_1, x_2, x_3, x_4 \} \) and let \( T : X \rightarrow X \) be a linear map such that

\[
T(x_1) = x_2, \quad T(x_2) = x_3, \quad T(x_3) = x_4, \quad T(x_4) = -x_1 - 4x_2 - 6x_3 - 4x_4.
\]

Is \( T \) diagonalizable over \( \mathbb{C} \)?

2. Prove that in a real Euclidean space, \( ||x|| = \max\{ (x, y) : ||y|| = 1 \} \).

3. Let \( f \) and \( g \) be continuous functions on the interval \([0, 1]\). Prove the following inequalities.

(a) \( \left( \int_0^1 f(t)g(t) \, dt \right)^2 \leq \int_0^1 f(t)^2 \, dt \int_0^1 g(t)^2 \, dt \)

(b) \( \left( \int_0^1 (f(t) + g(t))^2 \, dt \right)^{1/2} \leq \left( \int_0^1 f(t)^2 \, dt \right)^{1/2} + \left( \int_0^1 g(t)^2 \, dt \right)^{1/2} \).

4. Use the Gram-Schmidt process to find an orthonormal basis for the subspace of \( \mathbb{R}^4 \) spanned by \( y_1 = (1, 2, 1, 1) \), \( y_2 = (1, -1, 0, 2) \) and \( y_3 = (2, 0, 1, 1) \).

5. Let \( X \) be the vector space of all continuous real-valued functions on \([0, 1]\). Define an inner product on \( X \) by

\( (f, g) = \int_0^1 f(t)g(t) \, dt \).

Let \( Y \) be the subspace of \( X \) spanned by \( f_0, f_1, f_2, f_3 \), where \( f_k(x) = x^k \). Find an orthonormal basis for \( Y \).

6. Let \( Y \) be a subspace of a Euclidean space \( X \), and \( P_Y : X \rightarrow X \) the orthogonal projection onto \( Y \). Prove that \( P_Y^* = P_Y \).

7. Show that a matrix \( M \) is orthogonal iff its column vectors form an orthonormal set.

8. Let \( X \) be an \( n \)-dimensional real Euclidean space, and \( A : X \rightarrow X \) a linear map. Define the map \( f : X \rightarrow X \) by \( f(x, y) = x^T Ay \). Give (with proof) necessary and sufficient conditions on \( A \) for \( f \) to be an inner product on \( X \).