1. Consider the following matrix:

$$M_n = \begin{pmatrix} 0 & -a_0 \\ I_{n-1} & -\mathbf{a}_{n-1} \end{pmatrix} \quad \text{where } \mathbf{a}_n = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

   (a) Show that the characteristic polynomial of M_n is

   $$P_{M_n}(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \,.$$

   Here, I_{n-1} denotes the $(n-1)\times(n-1)$ identity matrix.

   (b) Is $P_{M_n}(t)$ also the minimal polynomial? Prove or disprove.

   (c) Now, let X be a 4 dimensional vector space over $\mathbb{R}$ with basis $\{x_1, x_2, x_3, x_4\}$ and let $T : X \to X$ be a linear map such that

   $$T(x_1) = x_2, \quad T(x_2) = x_3, \quad T(x_3) = x_4, \quad T(x_4) = -x_1 - 4x_2 - 6x_3 - 4x_4.$$

   Is T diagonalizable over $\mathbb{C}$?

2. Prove that in a real Euclidean space, $||x|| = \max\{(x, y) \colon ||y|| = 1\}$.

3. Let f and g be continuous functions on the interval $[0, 1]$. Prove the following inequalities.

   (a) $\left(\int_0^1 f(t)g(t)\,dt\right)^2 \leq \int_0^1 f(t)^2\,dt \int_0^1 g(t)^2 dt$

   (b) $\left(\int_0^1 (f(t)+g(t))^2\,dt\right)^{1/2} \leq \left(\int_0^1 f(t)^2\,dt\right)^{1/2} + \left(\int_0^1 g(t)^2 dt\right)^{1/2}.$

4. Use the Gram-Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $y_1 = (1, 2, 1, 1)$, $y_2 = (1, -1, 0, 2)$ and $y_3 = (2, 0, 1, 1)$.

5. Let X be the vector space of all continuous real-valued functions on $[0, 1]$. Define an inner product on X by

   $$(f, g) = \int_0^1 f(t)g(t)\,dt\,.$$

   Let Y be the subspace of X spanned by f_0, f_1, f_2, f_3, where $f_k(x) = x^k$. Find an orthonormal basis for Y.

6. Let Y be a subspace of a Euclidean space X, and $P_Y \colon X \to X$ the orthogonal projection onto Y. Prove that $P_Y^* = P_Y$.

7. Show that a matrix M is orthogonal iff its column vectors form an orthonormal set.

8. Let X be an n-dimensional real Euclidean space, and $A \colon X \to X$ a linear map. Define the map $f \colon X \to X$ by $f(x, y) = x^T Ay$. Give (with proof) necessary and suffcient conditions on A for f to be an inner product on X.