1. Let X be a finite-dimensional real Euclidean space. We say that a sequence $\{A_n\}$ of linear maps converges to a limit A if $\lim_{n\to\infty} ||A_n - A|| = 0$.

   (a) Show that $\{A_n\}$ converges to A if and only if for all $x \in X$, $A_n x$ converges to Ax.

   (b) Show by example that this fails if the dimension of X is infinite.

2. Let X be the space of continuous complex-valued functions on $[-1, 1]$ and define an inner product on X by

$$(f, g) = \int_{-1}^{1} f(s)\bar{g}(s)\, ds\,.$$

   Let $m(s)$ be a continuous function of absolute value 1, that is, $|m(s)| = 1$, $-1 \le s \le 1$. Define M to be multiplication by m:

$$(Mf)(s) = m(s)f(s)\,.$$

   Show that M is unitary.

3. Let A be a linear map of a finite-dimensional complex Euclidean space X.

   (a) Show that A is normal if and only if it unitarily similar to a diagonal matrix, that is, $A = U^*DU$ for a diagonal matrix D and unitary matrix U.

   (b) Prove that if A is normal then it has a square-root, that is, a matrix B such that $A = B^2$. Is B necessarily normal? Unique?

   (c) Suppose that A is diagonalizable. Prove that A is normal if and only if each eigenvector of A is an eigenvector of A^*.

4. Let A be a linear map of a finite-dimensional real Euclidean space X.

   (a) Show that if $A = P^{-1}DP$ for an orthogonal matrix P, then A is normal (that is, $AA^T = A^TA$).

   (b) Show by example that not all unitary matrices are orthogonal.

   (c) Show that not every normal matrix is orthogonally similar to a diagonal matrix.

5. Express $q(x_1, x_2, x_3) = 3x_1^2 + 8x_1x_2 - 7x_1x_3 + 12x_2^2 - 8x_2x_3 + 6x_3^2$ as $q(x) = x^TAx$, where A is symmetric.

6. Let

$$M = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

   and let $q(x) = (x, Mx)$. Find an orthogonal matrix P which diagonalizes the quadratic form q.

7. (a) Write the equation $5x_1^2 - 6x_1x_2 + 5x_2^2 = 1$ in the form $x^TAx = 1$.
   (b) Write $A = P^TDP$, where D is a diagonal matrix and P is orthogonal with determinant 1.
   (c) Sketch the graph of the equation $x^TDx = 1$ in the x_1x_2-plane.
   (d) Use a geometric argument applied to part (c) to sketch the graph of $x^TAx = 1$.
   (e) Repeat steps (a)–(d) for the equation $2x_1^2 + 6x_1x_2 + 2x_2^2 = 1$.

8. Let S be the cyclic shift mapping of $\mathbb{C}^n$, that is, $C(z_1, \ldots, z_n) = (z_n, z_1, \ldots, z_{n-1})$.
   (a) Prove that S is an isometry in the Euclidean norm.
   (b) Determine the eigenvalues and eigenvectors of S.
   (c) Verify that the eigenvectors are orthogonal.

   Hint: There are very short and elegant solutions for all three parts of this problem! You may find Problem 1 on HW 9 useful.