Throughout, $X$ is a finite-dimensional Euclidean space.

1. Define the *index* of a real symmetric matrix $A$ to be the number of strictly positive eigenvalues minus the number of strictly negative eigenvalues. Suppose $A$ and $B$ are real symmetric matrices and $x^T Ax \leq x^T B x$ for all $x \in X$. Prove that the index of $A$ is at most the index of $B$.

2. Let $H, M : X \to X$ be self-adjoint mappings, and $M$ positive definite. Define

$$R_{H,M}(x) = \frac{(x, Hx)}{(x, Mx)}.$$

(a) Let $\mu = \inf \{R_{H,M}(x) \mid x \in X\}$. Show that $\mu$ exists, and that there is some $v \in X$ for which $R_{H,M}(v) = \mu$, and that $\mu$ and $v$ satisfy $Hv = \mu Mv$.

(b) Show that the constrained minimum problem

$$\min \{R_{H,M}(y) \mid (y, Mv) = 0\}$$

has a nonzero solution $w \in X$, and that this solution satisfies $Hw = \kappa Mw$, where $\kappa = R_{H,M}(w)$.

3. Let $H, M : X \to X$ be self-adjoint mappings, and $M$ positive definite.

(a) Show that there exists a basis $v_1, \ldots, v_n$ of $X$ where each $v_i$ satisfies an equation of the form

$$Hv_i = \mu_i Mv_i \quad (\mu_i \text{ real}), \quad (v_i, Mv_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

(b) Compute $(v_i, Hv_j)$, and show that there is an invertible real matrix $U$ for which $U^* MU = I$ and $U^* H U$ is diagonal.

(c) Characterize the numbers $\mu_1, \ldots, \mu_n$ by a minimax principle.

4. Let $H, M : X \to X$ be self-adjoint mappings, and $M$ positive definite.

(a) Prove that all the eigenvalues of $M^{-1}H$ are real.

(b) Prove that if $H$ is positive-definite, then all the eigenvalues of $M^{-1}H$ are positive.

(c) What if $M$ is not positive definite?

5. Let $N : X \to X$ be a normal mapping of a Euclidean space. Prove that $\|N\| = \max |n_i|$, where the $n_i$s are the eigenvalues of $N$.

6. Let $A(t)$ be a matrix-valued function that is differentiable and invertible. Use the product rule $(\frac{d}{dt}[A(t)B(t)] = \dot{A}B + A\dot{B})$ to derive

$$\frac{d}{dt} A^{-1} = -A^{-1} \left( \frac{d}{dt} A \right) A^{-1}.$$