Throughout, X is a finite-dimensional Euclidean space.

1. Define the *index* of a real symmetric matrix A to be the number of strictly positive eigenvalues minus the number of strictly negative eigenvalues. Suppose A and B are real symmetric matrices and $x^T Ax \leq x^T Bx$ for all $x \in X$. Prove that the index of A is at most the index of B.

2. Let $H, M \colon X \to X$ be self-adjoint mappings, and M positive definite. Define

$$R_{H,M}(x) = \frac{(x, Hx)}{(x, Mx)}\,.$$

   (a) Let $\mu = \inf\{R_{H,M}(x) \mid x \in X\}$. Show that μ exists, and that there is some $v \in X$ for which $R_{H,M}(v) = \mu$, and that μ and v satisfy $Hv = \mu Mv$.

   (b) Show that the constrained minimum problem

$$\min\{R_{H,M}(y) \mid (y, Mv) = 0\}$$

   has a nonzero solution $w \in X$, and that this solution satisfies $Hw = \kappa Mw$, where $\kappa = R_{H,M}(w)$.

3. Let $H, M \colon X \to X$ be self-adjoint mappings, and M positive definite.

   (a) Show that there exists a basis $v_1, \dots, v_n$ of X where each v_i satisfies an equation of the form

$$Hv_i = \mu_i M v_i \quad (\mu_i \text{ real}), \qquad (v_i, Mv_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

   (b) Compute (v_i, Hv_j), and show that there is an invertible real matrix U for which $U^* MU = I$ and $U^* HU$ is diagonal.

   (c) Characterize the numbers $\mu_1, \dots \mu_n$ by a minimax principle.

4. Let $H, M \colon X \to X$ be self-adjoint mappings, and M positive definite.

   (a) Prove that all the eigenvalues of $M^{-1}H$ are real.

   (b) Prove that if H is positive-definite, then all the eigenvalues of $M^{-1}H$ are positive.

   (c) What if M is not positive definite?

5. Let $N \colon X \to X$ be a normal mapping of a Euclidean space. Prove that $||N|| = \max |n_i|$, where the n_is are the eigenvalues of N.

6. Let $A(t)$ be a matrix-valued function that is differentiable and invertible. Use the product rule ($\frac{d}{dt}[A(t)B(t)] = \dot{A}B + A\dot{B}$) to derive

$$\frac{d}{dt}A^{-1} = -A^{-1}\left(\frac{d}{dt}A\right)A^{-1}\,.$$