1. For each of the following sets, determine if it is a *vector space* over $\mathbb{R}$. If it is, give an explicit *basis* and compute its *dimension*. If it isn't, explain why not by giving an example of how one of the vector space axioms fails.

   (a) The set of points in $\mathbb{R}^3$ with $x = 0$.

   (b) The set of points in $\mathbb{R}^2$ with $x = y$.

   (c) The set of points in $\mathbb{R}^3$ with $x = y$.

   (d) The set of points in $\mathbb{R}^3$ with $z \geq 0$.

   (e) The plane in $\mathbb{R}^3$ defined by the equation $z = 2x - 3y$.

   (f) The set of unit vectors in $\mathbb{R}^2$.

   (g) The set of polynomials of degree n.

   (h) The set Poly_n of polynomials of degree at most n.

   (i) The set of polynomials of degree at most n, with only even-powers of x.

   (j) The set $\mathrm{Per}_{2\pi}$ of piecewise continuous functions $f(x)$ such that $f(x) = f(x + 2\pi)$ for all $x \in \mathbb{R}$.

   (k) $\mathbb{C} := \{a + bi \mid a, b \in \mathbb{R}\}$.

2. Let v_1, v_2, w be three *linearly independent* vectors in $\mathbb{R}^3$. That is, they do not all lie on the same plane. For each of the following (infinite) set of vectors, carefully sketch it in $\mathbb{R}^3$, and determine whether or not it is a vector space (i.e., a *subspace* of $\mathbb{R}^3$). Explain your reasoning.

   (a) $\{Cv_1 \mid C \in \mathbb{R}\}$

   (b) $\{Cv_1 + w \mid C \in \mathbb{R}\}$

   (c) $\{C_1v_1 + C_2v_2 \mid C_1, C_2 \in \mathbb{R}\}$

   (d) $\{C_1v_1 + C_2v_2 + w \mid C_1, C_2 \in \mathbb{R}\}$

3. Find the general solution to each of the following ODEs. Then, decide whether or not the set of solutions form a vector space. Explain your reasoning. Compare your answers to the previous problem.

   (a) $y' - 2y = 0$

   (b) $y' - 2y = 1$

   (c) $y'' + 4y = 0$

   (d) $y'' + 4y = e^{3t}$

4. Complete the following sentences as concisely as possible:

   (a) Two non-zero vectors $v_1, v_2 \in \mathbb{R}^2$ are a basis for $\mathbb{R}^2$ if and only if...

   (b) Three non-zero vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ are a basis for $\mathbb{R}^3$ if and only if...

5. For each of the following pairs of vectors $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$, carry out the following steps:

   i. Sketch v_1 and v_2 in the xy-plane.

   ii. The lines through v_1 and v_2 generate a grid (of parallelograms) on the xy-plane. Sketch this "grid," and compute the area of one of the parallelgrams.

   iii. Calculate the determinant of the matrix $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$. Is this matrix invertible?

   iv. Determine whether $\{v_1, v_2\}$ is a basis of $\mathbb{R}^2$.

   (a) $v_1 = (1, 0)$, $v_2 = (0, 1)$

   (b) $v_1 = (2, 0)$, $v_2 = (0, 2)$

   (c) $v_1 = (\frac{1}{2}, \frac{1}{2})$, $v_2 = (\frac{1}{2}, -\frac{1}{2})$

   (d) $v_1 = (1, 1)$, $v_2 = (1, 2)$

   (e) $v_1 = (1, 1)$, $v_2 = (-1, 3)$

   (f) $v_1 = (2, 1)$, $v_2 = (-4, 2)$

   Summarize your conclusions in a sentence or two.

6. Let v_1 and v_2 be two linearly independent vectors in $\mathbb{R}^3$, i.e., they determine a plane P.

   (a) Sketch the plane $P = \{C_1 v_1 + C_2 v_2 \mid C_1, C_2 \in \mathbb{R}\}$ in $\mathbb{R}^3$.

   (b) Sketch the vectors $w_1 := \frac{1}{2}v_1 + \frac{1}{2}v_2$ and $w_2 := \frac{1}{2}v_1 - \frac{1}{2}v_2$. Do these "determine" the same plane? In other words, is the following (infinite) set of vectors

   $$\left\{C_1\left(\tfrac{1}{2}v_1 + \tfrac{1}{2}v_2\right) + C_2\left(\tfrac{1}{2}v_1 - \tfrac{1}{2}v_2\right) \mid C_1, C_2 \in \mathbb{R}\right\}$$

   the same as $\{C_1 v_1 + C_2 v_2 \mid C_1, C_2 \in \mathbb{R}\}$?

   (c) Is $\{v_1, v_2\}$ a basis for P? Is $\{w_1, w_2\}$ a basis for P? Why or why not?

   (d) Calculate the determinant of the matrix $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$. Is this matrix invertible? How does this relate to the previous problem?

7. (a) Consider the ODE $y'' - \omega^2 y = 0$. If we assume that $y(t) = e^{rt}$ and plug this back in and solve for r, we get that $r = \pm\omega$. Therefore, the general solution is $y(t) = C_1 e^{\omega t} + C_2 e^{-\omega t}$, i.e., $\{e^{\omega t}, e^{-\omega t}\}$ is a *basis* for the solution space. Using the fact that $e^{\omega t} = \cosh \omega t + \sinh \omega t$, find a basis for the solution space involving hyperbolic trig functions, and write the general solution using these functions. *Hint*: Look at the previous problem!

   (b) Repeat the above exercise for the ODE $y'' + \omega^2 y = 0$. Specifically, if we assume that $y(t) = e^{rt}$, then we get that $r = \pm i\omega$. Therefore, the general solution is $y(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$, i.e., $\{e^{i\omega t}, e^{-i\omega t}\}$ is a basis for the solution space. Using Euler's formula: $e^{i\omega t} = \cos \omega t + i \sin \omega t$, find a basis for the solution space involving sine and cosines, and write the general solution using these functions.

8. For each of the following, a vector space V is given, along with a finite set $S \subset V$. Denote the subspace of V *spanned* by S as $\mathrm{Span}(S)$. Find an explicit basis for $\mathrm{Span}(S)$ and compute its dimension.

   (a) $V = \mathbb{R}^3$, $S = \{(1,0,0),(0,1,1),(1,1,1),(3,1,1)\}$.

   (b) $V = \mathbb{R}^2$, $S = \{(1,0),(0,1),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},-\frac{1}{2})\}$.

   (c) $V = \{\text{continuous functions } f\colon \mathbb{R} \to \mathbb{R}\}$, $S = \{e^{3x}, e^{-3x}, \cosh 3x, \sinh 3x\}$.

9. For each of the following triples of vectors $v_1 = (x_1,y_1,z_1)$, $v_2 = (x_2,y_2,z_2)$, and $v_3 = (x_3,y_3,z_3)$, carry out the following steps:

   i. Sketch v_1, v_2, and v_3 in $\mathbb{R}^3$.

   ii. Calculate the determinant of the matrix $\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$. Is this matrix invertible?

   iii. The lines through v_1, v_2, and v_3 generate a tessellation (of parallelepipeds) in $\mathbb{R}^3$. What do you think the volume of each parallelepiped is?

   iv. Describe in words (e.g., line, plane, all of $\mathbb{R}^3$) the subspace $\mathrm{Span}\{v_1,v_2,v_3\}$. Is $\{v_1,v_2,v_3\}$ is a basis of $\mathbb{R}^3$?

   (a) $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, $v_3 = (0,0,1)$

   (b) $v_1 = (2,0,0)$, $v_2 = (0,2,0)$, $v_3 = (0,0,2)$

   (c) $v_1 = (1,0,0)$, $v_2 = (0,1,1)$, $v_3 = (3,1,1)$

   (d) $v_1 = (1,0,0)$, $v_2 = (0,2,-1)$, $v_3 = (1,1,1)$

   Summarize your conclusions in a sentence or two.