1. Let $T\colon V \to W$ be a linear map between vector spaces. Prove that $\ker(T) := \{v \in V \mid T(v) = 0\}$ is a subspace of V.

2. Let $V = \mathcal{C}^1$, the vector space of differentiable real-valued functions. Consider the linear operator $T = \frac{d}{dt} + 3$.

   (a) The kernel of T can be characterized precisely by the set of all funtions that solve a particular differential equation. Write down this equation.

   (b) Find the general solution for the differential equation you found in Part (a), and hence an explicit formula for $\ker(T)$.

   (c) Write down an explicit basis for the solution space, $\ker(T)$. What is the dimension of this vector space?

3. Consider the line $x_2 + 3x_1 = 6$ in $\mathbb{R}^2$. In this problem, we will *parametrize* this line, by writing it as $\ell(t) = t\mathbf{v} + \mathbf{w}$, where $t \in (-\infty, \infty)$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are fixed.

   (a) Find a vector $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$ that lies on ℓ (*Any* particular vector will do!)

   (b) Set $(y_1, y_2) = (x_1 - w_1, x_2 - w_2)$, where (w_1, w_2) is the vector you found in Part (a). Solve for x_1 and x_2 in terms of y_1 and y_2, and plug them back into the orignal equation to get another equation for a line. This is called a *change of coordinates*, or *change of variables*. Note that this line goes through the origin in the (y_1, y_2)-coordinate system, i.e., it contains the vector $\mathbf{0} = (0, 0)$.

   (c) Write down a parameterization for the line in the (y_1, y_2)-coordinate system. Your answer will be of the form $m(t) = t\mathbf{v}$, where $\mathbf{v}$ is *any* non-zero vector on the line.

   (d) Write down a parameterization for the line in the (x_1, x_2)-coordinate system.

   (e) Sketch both lines, $\ell(t) = t\mathbf{v} + \mathbf{w}$ and $m(t) = t\mathbf{v}$, on the same set of axes.

4. Consider the differential equation $y' + 3y = 6$.

   (a) Find a particular solution $y_p(t)$ to this equation by inspection. *Hint:* Try $y_p(t) = c$, where c is a constant.

   (b) Set $y_h(t) = y(t) - y_p(t)$, where $y_p(t)$ is the function you found in Part (a). Plug this back into the original equation to get an ODE for $y_h(t)$. This is called a *homogeneous* ODE because $y_h(t) = 0$ is a solution.

   (c) Solve for $y_h(t)$. Compare your answer to Part (c) of the previous problem.

   (d) Solve for $y(t)$. Compare your answer to Part (d) of the previous problem.

5. Consider the plane P in $\mathbb{R}^3$ given by the equation $x_3 - 4x_1 = 5$. In this problem, we will parametrize P by writing it as $P(s,t) = s\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{w}$, where $s, t \in (-\infty, \infty)$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w} \in \mathbb{R}^3$ are fixed.

   (a) Find a vector $\mathbf{w} = (w_1, w_2, w_3)$ on P. (Any particular vector will do.)

   (b) Set $(y_1, y_2, y_3) = (x_1 - w_1, x_2 - w_2, x_3 - w_3)$. Change coordinates by plugging these values back into the equation for P.

(c) Parametrize the plane in this new coordinate system by writing $Q(s,t) = s\mathbf{v}_1 + t\mathbf{v}_2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are any two vectors on the plane that are not co-linear. Note that this plane goes through the origin in the (y_1, y_2, y_3)-coordinate system.

(d) Write down a parameterization for the plane in the (x_1, x_2, x_3)-coordinate system.

(e) Sketch both planes, $P(s,t) = s\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{w}$ and $Q(s,t) = s\mathbf{v}_1 + t\mathbf{v}_2$, on the same set of axes.

6. Let $\mathbf{v} = (3,4) \in \mathbb{R}^2$.

(a) Compute $||\mathbf{v}|| := \sqrt{\mathbf{v} \cdot \mathbf{v}}$.

(b) Recall that $\{\mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)\}$ is an *orthonormal basis* for $\mathbb{R}^2$. Decompose $\mathbf{v}$ into this basis, i.e., write $\mathbf{v} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ for some $a_1, a_2 \in \mathbb{R}$.

(c) Sketch $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{v}$ in $\mathbb{R}^2$. Graphically show what a_1 and a_2 represent in terms of the projection of $\mathbf{v}$ onto the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

(d) The set $\{\mathbf{v}_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), \mathbf{v}_2 = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\}$ is also an orthonormal basis. Decompose $\mathbf{v}$ into this basis, i.e., write $\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2$ for some $b_1, b_2 \in \mathbb{R}$.

(e) On a new set of axes, sketch $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}$ in $\mathbb{R}^2$. Graphically show what b_1 and b_2 represent in terms of the projection of $\mathbf{v}$ onto the unit vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

7. The following differential equations are linear but inhomogeneous. Carry out the following steps for each.

i. Find a particular solution, $y_p(t)$.

ii. Make a change of variables $y_h(t) = y(t) - y_p(t)$, and plug this back in to get a linear, homogeneous ODE in terms of y_h.

iii. The general solution is now $\ker(T)$, for some linear operation T. Find T.

iv. Solve for $y_h(t)$, and use this to find $y(t)$.

(a) $y' + 3y = 6$

(b) $y' + 3y = 2t + 4$

(c) $y' + 3y = 2e^{4t}$

(d) $y' + 3y = \cos 2t$

(e) $y'' + 4y = e^t$