1. (a) Consider an ODE
$$z'' + a(t)z' + b(t)z = 0\,,$$
and suppose that you know one complex-valued solution, $z(t) = x(t) + iy(t)$, where $x(t)$ and $y(t)$ are real-valued functions. Substitute $z(t)$ into the above equation and equate real and imaginary parts to show that $x(t)$ and $y(t)$ are also solutions. In what case do you have enough information to determine the general solution. What is the general solution in this case?

   (b) Now, consider the ODE $z'' + 6z' + 10z = 0$. Find a solution of the form $z_1(t) = e^{rt}$ for some r, and use Euler's equation, $e^{i\theta} = \cos\theta + i\sin\theta$, to break this solution into its real and imaginary parts.

   (c) Use the result of Part (a) to write the general solution.

2. Consider a 2nd order inhomogeneous ODE
$$y'' + a(t)y' + b(t)y = g(t)\,.$$

   (a) First, consider the associated homogeneous equation,
$$y_h'' + a(t)y_h' + b(t)y_h = 0.$$
Verify that the set of solutions is a vector space by showing that if $y_1(t)$ and $y_2(t)$ are any two solutions, then $C_1y_1(t) + C_2y_2(t)$ is a solution for any $C_1, C_2 \in \mathbb{R}$. [Note that this shows closure under addition and scalar multiplication at the same time!]

   (b) Now, consider the original inhomogeneous ODE, and suppose that $y(t)$ is the general solution and $y_p(t)$ is *any* one particular solution. Show by direct substitution that $y(t) - y_p(t)$ solves the homogeneous equation.

   (c) Conclude that $y(t) = y_h(t) + y_p(t)$, where $y_h(t)$ is the general solution to the related homogeneous equation. Given the extra knowledge that $y_h(t)$ is a 2-dimensional vector space, this allows us to conclude that $y(t) = C_1y_1(t) + C_2y_2(t) + y_p(t)$. Summarize this in a sentence (i.e., explain precisely what $y_1(t)$ and $y_2(t)$ are, because not just any two functions work).

3. Solve the following linear, homogeneous ODEs.

   (a) $y' + 3y = 0$

   (b) $4y'' + 4y' + 5y = 0$

   (c) $y'' - 6y' + 9y = 0$

   (d) $y'' + 5y' + 4y = 0$

4. Solve the following linear, inhomogeneous ODEs. Feel free to use the results of the previous problem without re-deriving them.

   (a) $y' + 3y = 2t^2$

(b) $4y'' + 4y' + 5y = e^{3t} + \sin 2t + 10$

(c) $y'' - 6y' + 9y = e^{2t}$

(d) $y'' + 5y' + 4y = e^{-t}$

5. If $y_f(t)$ is a solution of
$$y'' + a(t)y' + b(t)y = f(t)$$
and $y_g(t)$ is a solution of
$$y'' + a(t)y' + b(t)y = g(t)\,,$$
show that $z(t) = \alpha y_f(t) + \beta y_g(t)$ is a solution of
$$y'' + a(t)y' + b(t)y = \alpha f(t) + \beta g(t)\,,$$
where α and β are any real numbers, by plugging it into the ODE.

6. As we've seen, to solve ODE of the form
$$y'' + by' + cy = 0\,, \qquad b \text{ and } c \text{ constants}$$
we assume that the solution has the form e^{rt}, and then we plug this back into the ODE to get the *characteristic equation*: $r^2 + br + c = 0$. Given that this equation has a double root $r = r_1$ (i.e., the roots are $r_1 = r_2$), show by direct substitution that $y = te^{rt}$ is a solution of the ODE, and then write down the general solution.

7. Suppose a glass of water sitting in a room of constant temperature A obeys Newton's law of cooling. We can express this by the simple differential equation $T' = k(A - T)$, where $T(t)$ is the temperature of the water at time t.

(a) Solve the homogeneous equation, $T_h' = -kT_h$. What is the physical interpretation of $T_h(t)$?

(b) Find a particular solution $T_p(t)$ to the original equation using your physical intuition alone.

(c) Find the general solution by adding the solutions you got in Parts (a) and (b).

8. Suppose that the temperature of glass of water sitting outside obeys Newton's law of cooling. It is reasonable to expect that the ambient temperature is not constant, but rather a function $A(t)$ of time. Moreover, it is reasonable to assume that the ambient temperature is sinusoidal, so let $A(t) = B \sin \omega t$, where B is some constant. We now have the following equation that we wish to solve:
$$T' = k(B \sin \omega t - T)\,.$$

(a) Solve the related homogeneous equation, $T_h' = -kT_h$.

(b) To solve the ODE, we need to find any particular solution, so assume there is one of the form $T_p(t) = a \cos \omega t + b \sin \omega t$. Plug this back into the ODE, equate coefficients of the sine and cosine terms, and solve for a and b in terms of the amplitude B and the frequency ω.

(c) Find the general solution to this ODE.

(d) Give a qualitative physical description of what the particular solution $T_p(t)$ represents, and why. [Hint: Consider the long-term behavior of the temperature $T(t)$.]

9. Suppose we have a tank containing 150 gallons of solution with an unknown salt concentration. Suppose water with a salt concentration of 2 oz/gal flows in at a rate of 3 gal/min, and water (fully mixed) drains from the tank at the same rate. We wish to determine the salt content $x(t)$ at any point in time. Recall that we can do this by writing an ODE:

$$\begin{aligned} x'(t) &= (\text{rate in}) - (\text{rate out}) \\ &= (\text{flow rate in})(\text{incoming concentration}) - (\text{flow rate out})(\text{outgoing concentration}) \\ &= (3 \text{ gal/min})(2 \text{ oz/gal}) - (3 \text{ gal/min})\frac{x(t) \text{ oz}}{150 \text{ gal}} \\ &= 6 - \frac{1}{50}x(t)\,. \end{aligned}$$

All that is left is solving the ODE $x' = 6 - \frac{1}{50}x$ for $x(t)$.

(a) Sketch this physical situation.

(b) Solve the related homogeneous equation, $x_h' = -\frac{1}{50}x_h$.

(c) Find a simple particular solution $x_p(t)$ to the original ODE. Describe the physical situation that this represents.

(d) Find the general solution of this differential equation.