1. For each of the Cauchy-Euler equations, look for a solution of the form $y(t) = t^r$, and plug this back in to determine r. Find a basis of the solution space consisting of two real-valued functions, and use this to write the general solution.
   (a) $t^2y'' - ty' - 3y = 0$
   (b) $t^2y'' - ty' + 5y = 0$
   (c) $t^2y'' - 3ty' + 4y = 0$

2. Solve the following differential equations. You may freely use your answers to the previous problem without having to re-derive them.
   (a) $t^2y'' - ty' - 3y = 6$
   (b) $(t+2)^2y'' - (t+2)y' - 3y = 0$
   (c) $(t+2)^2y'' - (t+2)y' - 3y = 6$.

3. For each of the Cauchy-Euler equations, make the substitution $x = \ln t$ and write the resulting differential equation in $y(x)$ instead of $y(t)$. [No need to solve them.]
   (a) $t^2y'' - ty' - 3y = 0$
   (b) $t^2y'' - ty' + 5y = 0$
   (c) $t^2y'' - 3ty' + 4y = 0$

4. Write each of the following as a single series of the form $\sum f(n)t^n$. That is, $f(n)$ is the coefficient of t^n. You may need to additionally "pull out" the first term(s) from one of the sums.
   (a) $\displaystyle\sum_{n=0}^{5} t^{n-1}$
   (b) $\displaystyle\sum_{n=0}^{5} t^{n+1}$
   (c) $\displaystyle\sum_{n=0}^{\infty} na_nt^{n-1} + \sum_{n=0}^{\infty} a_nt^n$
   (d) $\displaystyle\sum_{n=0}^{\infty} a_nt^n + \sum_{n=0}^{\infty} b_nt^{n-1} + \sum_{n=0}^{\infty} c_nt^{n+1}$
   (e) $\displaystyle 5\sum_{n=0}^{\infty} n(n-1)t^{n-2} + 3\sum_{n=0}^{\infty} nt^{n-1} - 2\sum_{n=0}^{\infty} t^n$.

5. Consider the ODE $y'' - 2ty' + 10y = 0$. Note that unlike the equation in the first problem, there will not longer be a simple solution of the form t^r. However, we know that the solution space is 2-dimensional, and most "nice" functions can be written as a power series. Therefore, we'll look for a solution of the form $y(t) = \displaystyle\sum_{n=0}^{\infty} a_nt^n$.

(a) Plug $y(t)$ back into the ODE and find a recurrence relation for a_{n+2} in terms of a_n and a_{n+1}.

(b) Explicitly write out the coefficients a_n for $n \leq 9$, in terms of a_0 and a_1. Write down formulas for a_{2n} and a_{2n+1} in terms of a_0 and a_1.

(c) Since the solution space to this ODE is 2-dimensional, the general solution you found in Part (a) can be written as $y(t) = C_0 y_0(t) + C_1 y_1(t)$. Find such a *basis*, $\{y_0(t), y_1(t)\}$.

(d) Find a non-zero *polynomial* solution for this ODE. [*Hint*: Make a good choice for a_0 and a_1.]

(e) Are there any other polynomial solutions, excluding scalar multiples of the one you found in (d)? Why or why not?

(f) Consider the initial value problem

$$y'' - 2ty' + 10y = 0\,, \qquad y(0) = x_0\,, \quad y'(0) = v_0\,.$$

What are x_0 and v_0 in terms of the coefficients a_n?

6. The differential equation $(1-t^2)y'' - 2ty' + \lambda(\lambda+1)y = 0$, where λ is a constant, is known as *Legendre's equation*. It is used for modeling sperically symmetric potentials in the theory of Newtonian gravitation and in electricity and magnetism.

(a) Assume that the general solution has the form $y(t) = \sum_{n=0}^{\infty} a_n t^n$, and find the recursion formula for a_{n+2} in terms of a_n and a_{n+1}.

(b) Use the recursion formula to determine a_n in terms of a_0 and a_1, for $2 \leq n \leq 9$.

(c) For each $\lambda \in \mathbb{N}$, there will be a single (up to scalar multiples) nonzero polynomial solution $P_\lambda(t)$, called the *Legendre polynomial* of degree λ. Find the Legendre polynomial of degree $\lambda = 3$.

(d) Find a basis for the solution space to $(1-t^2)y'' - 2ty' + 12y = 0$.

7. The differential equation $y'' - ty = 0$ is called *Airy's equation*, and is used in physics to model the refraction of light.

(a) Assume the solution is a power series, and find the recurrence relation of the coefficients. [*Hint*: When shifting the indices, one way is to let $m = n-3$, then factor out t^{n+1} and find a_{n+3} in terms of a_n. Alternatively, you can find a_{n+2} in terms of a_{n-1}.]

(b) Show that $a_2 = 0$. [*Hint*: the two series for y'' and ty don't "start" at the same power of t, but for any solution, each term must be zero. (Why?)]

(c) Find the particular solution when $y(0) = 1$, $y'(0) = 0$, as well as the particular solution when $y(0) = 0$, $y'(0) = 1$.