1. For this problem, consider the vector space $V = \mathbb{R}^3$ and use the vector dot product as the inner product.

   (a) Show that the set of vectors $\{v_1, v_2, v_3\}$, where

   $$v_1 = (1, 2, -2)\,, \qquad v_2 = (0, 1, 1)\,, \qquad v_1 = (-4, 1, -1)\,.$$

   is an orthogonal set, but not orthonormal.

   (b) Normalize v_1, v_2, and v_3 to get an *orthonormal* basis of $\mathbb{R}^3$. That is, compute the following:

   $$\mathcal{B} = \{n_1, n_2, n_3\} \qquad \text{where} \quad n_i = \frac{v_i}{||v_i||}\,.$$

   (c) Use the dot product to express the vector $w = (1, 2, 3)$ in terms of n_1, n_2, and n_3. That is, find C_1, C_2, and C_3 such that

   $$w = C_1 n_1 + C_2 n_2 + C_3 n_3\,.$$

2. Let $\mathrm{Poly}_3(\mathbb{R}) = \{a_3x^3 + a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R}\}$, the vector space of polynomials of degree at most 3. Define the following *inner product* on $\mathrm{Poly}_3(\mathbb{R})$:

   $$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx\,.$$

   (a) Verify that this is indeed an inner product on $\mathrm{Poly}_3(\mathbb{R})$.

   (b) Consider the two sets

   $$\mathcal{B}_1 = \{1, x, 3x^2 - 1, 5x^3 - 3x\}\,, \qquad \mathcal{B}_2 = \{1, x, x^2, x^3\}$$

   that are both bases for $\mathrm{Poly}_3(\mathbb{R})$. Show that $\mathcal{B}_1$ is an orthogonal set, but $\mathcal{B}_2$ is not. (The set $\mathcal{B}_1$ are the first four *Legendre polynomials*, $P_n(x)$ for $n = 0, \ldots, 3$. When we study Sturm-Liouville theory, we will see why the Legendre polynomials are always orthogonal!)

   (c) For each $f \in \mathcal{B}_1$, compute the *norm* of f, which is defined as $||f|| = \langle f, f\rangle^{1/2}$. Find an *orthonormal* basis for $\mathrm{Poly}_3(\mathbb{R})$ by normalizing the elements in $\mathcal{B}_1$.

   (d) Consider the polynomial $f(x) = 3x^3 - 2x^2 + 4$. Use orthogonality to write $f(x)$ using the elements in $\mathcal{B}_1$. That is, find C_0, C_1, C_2, and C_3 such that

   $$3x^3 - 2x^2 + 4 = C_0 + C_1x + C_2(3x^2 - 1) + C_3(5x^3 - 3x).$$

3. Find the Fourier series of the following functions *without* computing any integrals.

   (a) $f(x) = 2 - 3\sin 4x + 5\cos 6x$,

   (b) $f(x) = \sin^2 x$. [*Hint*: Use a standard trig identity.]

4. Consider the 2π-periodic function defined by

$$f(x) = \begin{cases} x^2 & -\pi \leq x < \pi, \\ f(x-2k\pi) & -\pi + 2k\pi \leq x < \pi + 2k\pi. \end{cases}$$

Sketch this function on $[-7\pi, 7\pi]$ and compute its Fourier series. Feel free to use a computer to evaluate the indefinite integral $\int x^2 \cos nx \, dx$.

5. Let $\mathrm{Per}_{2\pi}(\mathbb{C})$ be the set of piecewise continuous 2π-periodic complex-valued functions, and define the (complex) inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-pi}^{\pi} f(x)\overline{g(x)} \, dx \,,$$

where $\overline{g(x)}$ denotes the complex conjugate of $g(x)$.

(a) Verify that this is indeed a complex inner product on $\mathrm{Per}_{2\pi}(\mathbb{C})$.

(b) Verify that $\mathcal{B} = \{e^{inx} \mid n \in \mathbb{Z}\}$ is an orthonormal set with respect to this inner product. That is, show that for each $f, g \in \mathcal{B}$, we have $\langle f, g \rangle = 1$ if $f = g$ and 0 if $f \neq g$. [*Hint*: Recall that $\overline{e^{inx}} = e^{-inx}$.]

(c) Given the additional knowledge that $\mathcal{B}$ is an orthonormal *basis* of $\mathrm{Per}_{2\pi}(\mathbb{C})$ (you may assume this), we can write any $f \in \mathrm{Per}_{2\pi}(\mathbb{C})$ uniquely as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \,, \qquad c_n \in \mathbb{C} \,.$$

Use the results from Parts (a) and (b) to derive a formula for each c_n. [*Hint*: Think of c_n as the magnitude of the "projection" of $f(x)$ onto e^{inx}.]