In the process of solving these problems, you will encounter several ODEs, Sturm-Liouville problems, and Fourier series, many of which you have encountered before (especially on the previous homework). You do not need to re-derive the solutions of anything you have previously solved.

1. We will solve for the function $u(x,t)$, defined for $0 \le x \le 1$ and $t \ge 0$, which satisfies the following conditions:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = 5\sin \pi x + 3 \sin 2\pi x.$$

   This PDE is called the *heat equation*. It is homogeneous.

   (a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of both boundary conditions and the initial condition.

   (b) Assume that $u(x,t) = f(x)g(t)$. Find u_t and u_{xx}. Also, determine the boundary conditions for $f(x)$ from the boundary conditions for $u(x,t)$.

   (c) Plug $u = fg$ back into the PDE and divide both sides by $c^2 fg$, i.e., "separate variables." Briefly justify why this quantity must be a constant. Call this constant $-\lambda$. Write down two ODEs: One for $g(t)$, and a Sturm-Lioville problem for $f(x)$.

   (d) Determine $f(x)$, λ, and $g(t)$.

   (e) Using your solution to Part (d) and linearity (in physical terms, the principle of superposition), find the general solution to this PDE.

   (f) Solve the initial value problem. That is, find the particular solution $u(x,t)$ that satisfies the boundary and initial conditions.

   (g) What is the steady-state solution, i.e., $\lim_{t\to\infty} u(x,t)$? Explain the physical interpretation of this solution.

2. Consider the following partial differential equation:

$$u_t = c^2 u_{xx}\,, \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = x(1-x)\,.$$

   The only difference between this and the previous equation is the *initial condition*. Carry out each step (a)–(g) from the previous problem, unless it is identical, in which case you should say "same as the previous problem" instead.

3. Consider the following partial differential equation:

$$u_t = c^2 u_{xx}\,, \qquad u_x(0,t) = u_x(1,t) = 0, \qquad u(x,0) = x(1-x)\,.$$

   The only difference between this and the previous equation are the *boundary conditions*. Carry out each step (a)–(g) from the previous problem, unless it is identical, in which case you should say "same as the previous problem" instead.

4. Solve the following PDE:

$$u_t = c^2 u_{xx}\,, \qquad u(0,t) = u_x(\pi,t) = 0, \qquad u(x,0) = 6 \sin \frac{5x}{2}\,.$$

   Make sure you describe and sketch the physical situation which this models, and determine the steady-state solution. Also, make it clear what Sturm-Liouville problem arises.

5. Consider the following *inhomogeneous* partial differential equation:

$$u_t = c^2 u_{xx}\,, \qquad u(0,t) = 32\,, \quad u(1,t) = 42\,, \qquad u(x,0) = x(1-x) + (10x+32)\,.$$

   (a) Make the substitution $v(x,t) = u(x,t) - (10x+32)$, and re-write the PDE above, including the boundary and initial conditions, in terms of v instead of u.

   (b) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of both boundary conditions and the initial condition.

   (c) Solve the PDE for $u(x,t)$. You may (and should!) use the results of a previous problem.

   (d) What is the steady-state solution? Give a physical interpretation of this.

6. Consider the following partial differential equation, subject to *periodic boundary conditions*:

$$u_t = c^2 u_{\theta\theta}, \qquad u(\theta+2\pi, t) = u(\theta, t), \qquad u(\theta, 0) = 2 + 4\sin 3\theta - \cos 5\theta.$$

   Briefly describe, and sketch, a physical situation which this models. [*Hint*: $u(\theta,t)$ is a function of an *angle* θ, and time t.] Solve this PDE via steps (a)–(g) from the first problem. You do not need to repeat any steps that are identical so something you've done before, as long as you point that out. However, you have *not* seen this type of Sturm-Liouville problem before. The boundary conditions are not specified, but rather are forced to be periodic.