In the process of solving these problems, you will encounter several ODEs, Sturm-Liouville problems, and Fourier series, many of which you have encountered before. You do not need to re-derive the solutions of anything you have previously solved.

1. Let $u(x,t)$ be the temperature of a bar of length 10, at position x and time t (in hours). Suppose that the left endpoint of the bar is not insulated, but the right endpoint is fully insulated, and the bar is sitting in a 70° room. Moreover, suppose that initially, the temperature increases linearly from 70° at the left endpoint to 80° at the other end. Finally, suppose the interior of the bar is poorly insulated, so heat can escape.

   (a) Suppose that heat escapes at a constant rate of 1° per hour. Write (but do not solve) an initial/boundary value problem for $u(x,t)$ that could model this situation.

   (b) A more realistic situation would be for heat to escape not at a constant rate, but at a rate proportional to the *difference* between the temperature of the bar and the ambient temperature of the room. Write an initial/boundary value problem for $u(x,t)$ that could model this situation. What is the steady-state solution and why?

2. Consider the following PDE:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = 0, \quad u_x(1,t) = \gamma\, u(1,t), \qquad u(x,0) = h(x)\,,$$

   where $\gamma \geq 0$ is a constant and $h(x)$ an arbitrary function satisfying $h(0) = 0$ and $h'(1) = \gamma\, h(1)$.

   (a) Describe a physical situation that this models. Be sure to describe the impact of the initial condition, *both* boundary conditions, and the constant γ. [*Hint*: First consider the special case when $\gamma = 0$. How does this compare to when $\gamma \approx 0$?]

   (b) What is the steady-state solution, and why? (Use your physical intuition.)

   (c) Assume that $u(x,t) = f(x)g(t)$, and plug this back in to get a familiar ODE for $g(t)$, and a Sturm-Liouville problem for $f(x)$.

   (d) Solve the Sturm-Liouville problem. You won't get a closed form for the eigenvalues, but you can write them as $\lambda_n = \omega_n^2$, where $\{\omega_n \mid n = 1, 2, \dots\}$ are the non-negative solutions to a simple algebraic equation. Write down this equation and graph it, clearly marking the first few solutions on the graph.

   (e) Find the general solution of this PDE, which will have the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n f_n(x) g_n(t),$$

   where $f_n(x)$ is the eigenfunction associated with eigenvalue λ_n.

   (f) The particular solution satisfying the initial and boundary conditions can be found by plugging in $t = 0$ and using the initial condition to determine the b_n. You do not need to do this, but explain the process in a few steps in enough detail that an educated reader (e.g., one of your classmates) could figure out how to do this from your explanation. Your answer should involve the orthogonality of the eigenfunctions and the associated inner product.

3. We will solve for the function $u(x,t)$, defined for $0 \leq x \leq 1$ and $t \geq 0$, which satisfies the following conditions:

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = x(1-x), \quad u_t(x,0) = 0.$$

This PDE is called the *wave equation*.

(a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of both boundary conditions and both initial conditions.

(b) Assume that $u(x,t) = f(x)g(t)$. Find u_t and u_{xx}. Also, determine the boundary conditions for $f(x)$ from the boundary conditions for $u(x,t)$, and one initial condition for $g(t)$. (The three BC/ICs equal to zero will give you these.)

(c) Plug $u = fg$ back into the PDE and divide both sides by $c^2 fg$, i.e., "separate variables." Briefly justify why this quantity must be a constant. Call this constant $-\lambda$. Write down two ODEs: One for $g(t)$ with one initial condition, and a Sturm-Lioville problem for $f(x)$.

(d) Determine $f(x)$, λ, and $g(t)$.

(e) Using your solution to Part (d) and the principle of superposition, find the general solution to this PDE.

(f) Use the last initial condition, $u(x,0) = x(1-x)$, to find the particular solution that satisfies the initial and boundary conditions.

(g) What is the long-term behavior of this system? Give both a mathematical and physical justification.

4. We will solve for the function $u(x,t)$, defined for $0 \leq x \leq 1$ and $t \geq 0$, which satisfies the following conditions:

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = 0, \quad u_t(x,0) = x(1-x).$$

The only difference between this and the previous equation are the initial conditions. Carry out each step (a)–(g) from the previous problem, unless it is idential (many of them will be), in which case you should say "same as the previous problem" instead.

5. Let $u(x,t)$ be defined for $0 \leq x \leq 1$ and $t > 0$, and consider the following PDE

$$u_{tt} + 2\beta u_t = u_{xx}, \qquad u(0,t) = u(1,t) = 0, \quad u(x,0) = x(1-x), \quad u_t(x,0) = 1.$$

where $0 < \beta < 1$ is a constant. This is the wave equation, where the transverse vibrations take place in a medium that imparts a resistance proportional to the instantaneous velocity.

(a) Describe and sketch this situation at $t = 0$.

(b) Assume that there is a solution of the form $u(x,t) = f(x)g(t)$. Plug this back into the PDE and get an ODE for $g(t)$ and a Sturm-Liouville problem for $f(x)$.

(c) The Sturm-Liouville problem should be familiar: $f'' = -\lambda f$, $f(0) = f(1) = 0$, and we've seen that the eigenvalues are $\lambda_n = n^2$ for $n = 1, 2, \ldots$ and the eigenfunctions are $f_n(x) = b_n \sin(n\pi x)$. The equation for $g(t)$ should be $g'' + 2\beta g' + n^2 g = 0$. Solve this ODE for $g(t)$.

(d) Write down the general solution to this PDE.

(e) Use the initial conditions to find the particular solution solving the boundary and initial conditions. (Feel free to use WolframAlpha to compute any derivatives you may need.)

(f) What is the long-term behavior of this system? Give both a mathematical and physical justification.