In the process of solving these problems, you will encounter several ODEs, Sturm-Liouville problems, PDEs, and Fourier series, many of which you have encountered before. You do not need to re-derive the solutions of anything you have previously solved.

1. For each of the following functions, compute its *Laplacian*, $\nabla^2 f$, and determine if it is *harmonic*. Feel free to use WolframAlpha to compute each individual second-partial derivative, as that is just a calculus exercise.

   (a) $f(x) = 10 - 3x$.

   (b) $f(x, y) = x^2 + y^2$.

   (c) $f(x, y) = e^x \cos y$.

   (d) $f(x, y) = x^3 - 3xy^2$.

   (e) $f(x, y) = \ln(x^2 + y^2)$.

   (f) $f(x, y, z) = \ln(x^2 + y^2 + z^2)$.

   (g) $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$.

2. Let $u(x, y)$ be a function defined on a square region, $0 \leq x, y \leq 1$. In this problem, we will solve *Laplace's equation* in this region subject to different *Dirichlet* boundary conditions.

   (a) Solve the following boundary value problem:

   $$\nabla^2 u = 0, \qquad u(0, y) = u(1, y) = u(x, 0) = 0, \quad u(x, 1) = 4 \sin \pi x - 3 \sin 2\pi x + 2 \sin 3\pi x.$$

   (b) Solve the following boundary value problem:

   $$\nabla^2 u = 0, \qquad u(x, 0) = u(x, 1) = u(0, y) = 0, \quad u(1, y) = y(1 - y).$$

   (c) Solve the following boundary value problem:

   $$\nabla^2 u = 0, \qquad u(x, 0) = u(0, y) = 0, \quad \begin{array}{l} u(x, 1) = 4 \sin \pi x - 3 \sin 2\pi x + 2 \sin 3\pi x, \\ u(1, y) = y(1 - y). \end{array}$$

   Hint: Use superposition!

   (d) Sketch the solutions to (a), (b), and (c). Feel free to use WolframAlpha to help.

3. Let $f(x, y)$ be a function defined on a square region, $0 \leq x, y \leq 1$. Find the general solution to the following PDE, called the *Helmholtz equation*, subject to the given (Dirichlet) boundary conditions:

   $$\nabla^2 f = -\lambda f, \qquad f(x, 0) = f(x, 1) = f(0, y) = f(1, y) = 0.$$

   The solutions to this PDE can be thought of as *eigenfunctions* of the Laplace operator (in rectangular coordinates) and the values of λ are the corresponding (Dirchlet) eigenvaules.

4. Let $u(x,y,t)$ be a function defined on a square region, $0 \leq x, y \leq 1$ and $t \geq 0$. Consider the following initial/boundary value problem for the *two-dimensional heat equation*:

$$u_t = c^2 \nabla^2 u, \qquad \begin{aligned} &u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0, \\ &u(x,y,0) = 2\sin \pi x \sin \pi y + 5 \sin 2\pi x \sin \pi y. \end{aligned}$$

   (a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of the boundary conditions and the initial condition.

   (b) Assume that there is a solution of the form $u(x,y,t) = f(x,y)g(t)$. Find u_{xx}, u_{yy}, and u_t.

   (c) Plug $u = fg$ back into the PDE and "separate variables" by dividing both sides by $c^2 fg$. Briefly justify why this quantity must be a constant. Call this constant $-\lambda$.

   (d) Write down, and then solve, an ODE for $g(t)$, and a PDE for $f(x,y)$ with four boundary conditions.

   (e) Find the general solution to this boundary value problem.

   (f) Find the particular solution that additionally satisfies the initial condition.

   (g) What is the steady-state solution? Give a mathematical and intuitive (physical) justification.

5. Consider the following initial/boundary value problem for the heat equation in a square region, and the funtion $u(x,y,t)$, where $0 \leq x, y \leq 1$ and $t \geq 0$.

$$u_t = c^2 \nabla^2 u, \qquad \begin{aligned} &u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0, \\ &u(x,y,0) = (3\sin \pi x + 5 \sin 2\pi x)\, y(1-y). \end{aligned}$$

   Since the only difference between this problem and the previous one is in the initial condition, steps (b)–(e) are the same and need not be repeated. Carry out steps (f) and (g), given this new initial condition.

6. Consider the following boundary value problem for the heat equation in a square region, $0 \leq x, y \leq 1$ and $t \geq 0$:

$$u_t = c^2 \nabla^2 u, \qquad \begin{aligned} &u(x,0,t) = u(0,y,t) = 0 \\ &u(x,1,t) = 4\sin \pi x - 3 \sin 2\pi x + 2 \sin 3\pi x \\ &u(1,y,t) = y(1-y). \end{aligned} \qquad u(x,y,0) = 1 + h(x,y)$$

   Here, $h(x,y)$ denotes the *steady-state* solution.

   (a) What is $h(x,y)$? Justify your answer. [*Hint*: This can be found by setting $u_t = 0$. Use the results of a previous problem.]

   (b) Make the substitution $v(x,y,t) = u(x,y,t) - h(x,y)$ and re-write the PDE above, including the boundary and initial conditions, in terms of v instead of u. (The resulting PDE is *homogeneous* – you should recognize it from a previous problem.)

   (c) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of the boundary conditions and the initial condition, as well as the difference between what u and v represent.

(d) Find the general solution for $u(x, y, t)$. [*Hint*: Observe that u is just the sum of the steady-state solution and the general solution of the related homogeneous boundary value problem.]

(e) Find the particular solution that additionally satisfies the initial condition.

7. Consider the following initial value problem for the wave equation in a square region, where $0 \leq x, y \leq 1$ and $t \geq 0$:

$$u_{tt} = c^2 \nabla^2 u, \qquad \begin{aligned} &u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0, \\ &u(x,y,0) = x(1-x)y(1-y), \\ &u_t(x,y,0) = 1. \end{aligned}$$

(a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of the boundary conditions and both initial conditions.

(b) Solve this PDE. The major difference between this and the heat equation is in the function $g(t)$. You do not need to repeat steps that are identical to previous problems.

(c) What is the long-term behavior of this system? Give a mathematical and intuitive (physical) justification.