2. Linear differential equations.

Def: A 1st order differential equation (ODE) is of the form $y' = f(t, y)$

A 2nd order differential equation is of the form $y'' = f(t, y, y')$.

Higher order ODE's are defined similarly.

* We are mainly concerned with ODE's that are "linear", as these are the most common.

Def: A 1st order ODE is linear if it can be written as

$y' + a(t)\, y = g(t)$. It is homogeneous if $g(t) = 0$.

A 2nd order ODE is linear if it can be written as

$y'' + a(t)\, y' + b(t)\, y = g(t)$. It is homogeneous if $g(t) = 0$.

Motivation for this terminology:

Let $y'' + a(t)\, y' + b(t)\, y = g(t)$ be a 2nd order ODE. Then

$T = \frac{d^2}{dt^2} + a(t) \frac{d}{dt} + b(t)$ is a linear operator on the space of (infinitely) differentiable functions, $\mathcal{C}^\infty$. Precisely:

$$T(y) = \left(\frac{d^2}{dt^2} + a(t) \frac{d}{dt} + b(t) \right) y = y'' + a(t)\, y' + b(t)\, y.$$

The kernel of this operator is the set of all solutions to the "related homogeneous ODE", $y'' + a(t)\, y' + b(t)\, y = 0$.

* <u>Fundamental theorem</u>: Let $T: \mathcal{C}^\infty \to \mathcal{C}^\infty$ be a linear differential operator of order n. Then $\ker T$ is an <u>n-dimensional subspace</u> of $\mathcal{C}^\infty$.

<u>What this means:</u>

- The solution to a linear, homogeneous ODE $y' + a(t)\,y = 0$ has the form $\ker\left(\frac{d}{dt} + a(t)\right) = \{C_1 y_1(t) \mid C_1 \in \mathbb{R}\}$. Here, $\{y_1(t)\}$ is a <u>basis</u> of the "solution space."
- The solution to a linear homogeneous ODE $y'' + a(t)\,y' + b(t)\,y = 0$ has the form $\ker\left(\frac{d^2}{dt^2} + a(t)\frac{d}{dt} + b(t)\right) = \{C_1 y_1(t) + C_2 y_2(t) \mid C_1, C_2 \in \mathbb{R}\}$.

  Here, $\{y_1(t), y_2(t)\}$ is a <u>basis</u> of the solution space

It should be clear how this extends to ODE's of order n.

<u>Big idea</u>: To solve an n^{th} order linear homogeneous ODE, we <u>need</u> to (somehow) find n linearly independent solutions, i.e., a basis for the solution space.

Let's recall how to do this:

$\boxed{1^{st}\text{ order}}$ Solve $y' + a(t)\,y = 0$.

My approaches (in order): (i) Inspection (e.g., if $a(t) = k$).

(ii) Separate variables

Examples:

(i) Solve $y' + ky = 0$. By inspection, $y(t) = Ce^{-kt}$

(ii) Solve $y' + ty = 0$.

always keep constants on this side.

Separate variables: $\frac{dy}{dt} = -ty \Rightarrow \int \frac{dy}{y} = -\int t \, dt$

$\ln y = -\frac{1}{2}t^2 + C$ (think: Why don't we need $\ln|y|$.)

$y = e^{-\frac{1}{2}t^2 + c} = Ce^{-\frac{1}{2}t^2}$.

Remark: This always works, assuming we can evaluate $\int a(t)\, dt$.

$\boxed{2^{nd} \text{ order}}$ Solve $y'' + a(t)\, y' + b(t)\, y = 0$.

This can be very hard, or unrealistic for arbitrary functions $a(t)$, $b(t)$. But if $a(t)$ & $b(t)$ are constants (a common case, e.g., mass-spring systems, RLC circuits, etc.), we can do it.

Suppose $y'' + py' + qy = 0$ (p, q constant)

Examples:

(i) Solve $y'' + 3y' + 2y = 0$.

Guess: $y(t) = e^{rt}$ is a solution. (Why might this work?)

Plug back in: $y' = re^{rt}$, $y'' = r^2e^{rt}$.

4

$r^2 e^{rt} + 3r e^{rt} + 2e^{rt} = 0$

$\cancel{e^{rt}}(r^2 + 3r + 2) = 0$ (Note: This would not work if it $\neq 0$.)

$(r+1)(r+2) = 0 \implies r_1 = -1, \quad r_2 = -2.$

We have found two linearly independent sol'ns: $y_1(t) = e^{-t}$ and $y_2(t) = e^{-2t}$.

Span $\{y_1(t), y_2(t)\}$ is a 2-dim'l subspace of the 2-dim'l solution space, so it must be the whole space, i.e., the general solution is $\{C_1 y_1(t) + C_2 y_2(t) \mid C_1, C_2 \in \mathbb{R}\}$.

(ii) Solve $y'' - 4y' + 20y = 0$.

Guess: $y(t) = e^{rt}, \quad y' = r e^{rt}, \quad y'' = r^2 e^{rt}$

Plug back in: $\cancel{e^{rt}}(r^2 - 4r + 20) = 0$

Quadratic eq'n $\implies r_{1,2} = 2 \pm 4i$.

We have found two linearly independent sol'ns: $y_1(t) = e^{(2+4i)t}$ and $y_2(t) = e^{(2-4i)t}$, and so $\{y_1(t), y_2(t)\}$ is a <u>basis</u> of the solution space, $\{C_1 y_1(t) + C_2 y_2(t) \mid C_1, C_2 \in \mathbb{C}\}$.

We can write an arbitrary sol'n as $y(t) = C_1 e^{(2+4i)t} + C_2 e^{(2-4i)t}$.

However, we don't "like" writing solutions using complex numbers

Note: $y_1(t) = e^{(2+4i)t} = e^{2t}e^{4it} = e^{2t}(\cos 4t + i \sin 4t)$

$y_2(t) = e^{(2-4i)t} = e^{2t}e^{-4it} = e^{2t}(\cos 4t - i \sin 4t)$.

Also, $\frac{1}{2}y_1(t) + \frac{1}{2}y_2(t) = e^{2t}\cos 4t$ is a sol'n

$\frac{1}{2i}y_1(t) + \frac{1}{2i}y_2(t) = e^{2t}\sin 4t$ is a sol'n.

These functions are linearly indepenent.

Thus, $\text{Span}\{e^{(2+4i)t}, e^{(2-4i)t}\} = \text{Span}\{e^{2t}\cos 4t, e^{2t}\sin 4t\}$.

So, another way to write the solution space is as

$\{C_1 e^{2t}\cos 4t + C_2 e^{2t}\sin 4t\}$, which (if we allow $C_1, C_2 \in \mathbb{C}$)

is exactly the same set of functions as we had before!

We say that the "general solution" is $y(t) = C_1 e^{2t}\cos 4t + C_2 e^{2t}\sin 4t$.

(iii) Solve $y'' + 4y' + 4y = 0$.

Guess: $y(t) = e^{rt}$, $y' = re^{rt}$, $y'' = r^2 e^{rt}$

Plug back in: $\cancel{e^{rt}}(r^2 + 4r + 4) = 0 \Rightarrow r_1 = r_2 = -2$.

Thus, $y_1(t) = e^{-2t}$ is a sol'n.

Problem: The sol'n space is 2-dimensional. We've only found one solution. We need one more.

Remark: Any sol'n $y_2(t)$ will do, as long as $y_2(t) \neq Ce^{-2t}$.

<u>Guess</u>: $y(t) = v(t)\,e^{-2t}$, solve for $v(t)$.

$$y' = -2e^{-2t}v + e^{-2t}v', \qquad y'' = -2(-2e^{-2t}v + e^{-2t}v') + (-2e^{-2t}v' + e^{-2t}v'')$$
$$= 4e^{-2t}v - 4e^{-2t}v' + e^{-2t}v''.$$

Plug back in: $(4e^{-2t}v - 4e^{-2t}v' + e^{-2t}v'') + 4(-2e^{-2t}v + e^{-2t}v') + 4e^{-2t}v = 0$

$$\Rightarrow e^{-2t}v'' = 0 \quad \Rightarrow v'' = 0 \quad \Rightarrow v(t) = at + b.$$

Thus, $y(t) = (at+b)e^{-2t}$ is a sol'n for <u>any</u> $a, b \in \mathbb{R}$.

We only need to pick one (not of the form Ce^{-2t}).

Take $a=1$, $b=0$ to get $y_2(t) = te^{-2t}$.

We now have a <u>basis</u> for our solution space: $\{e^{-2t}, te^{-2t}\}$,

so the sol'n space is $\mathrm{Span}\{e^{-2t}, te^{-2t}\} = \{C_1e^{-2t} + C_2te^{-2t} \mid C_1, C_2 \in \mathbb{R}\}$.

We say that the "general sol'n" is $y(t) = C_1e^{-2t} + C_2te^{-2t}$.

* <u>Summary so far</u>: We've seen how to solve (some) linear, homogeneous ODEs. The solutions sets are a <u>vector spaces</u>.

Now, let's look at how to solve linear, <u>inhomogeneous</u> equations.

$\boxed{1^{st}\text{ order}}$ Consider $y' + a(t)y = g(t)$, $g(t) \neq 0$.

Define the <u>related homogeneous equation</u> to be $y_h' + a(t)y_h = 0$.

<u>Theorem</u>: Let $y(t)$ be the general solution to $y' + a(t)\, y = g(t)$, and let $y_h(t)$ be the general solution to the related homogeneous equation, $y_h' + a(t)\, y_h = 0$. (so $y_h(t) = C y_1(t)$ for some $y_1(t)$.) Then if $y_p(t)$ is <u>any</u> particular sol'n to the original ODE, $\boxed{y(t) = y_h(t) + y_p(t)}$.

<u>Proof</u>: We'll show that $y(t) - y_p(t)$ solves the homogeneous eq'n.

Plug back in: $(y - y_p)' + a(t)(y - y_p) = (y' - a(t)\, y) - (y_p' - a(t)\, y_p)$

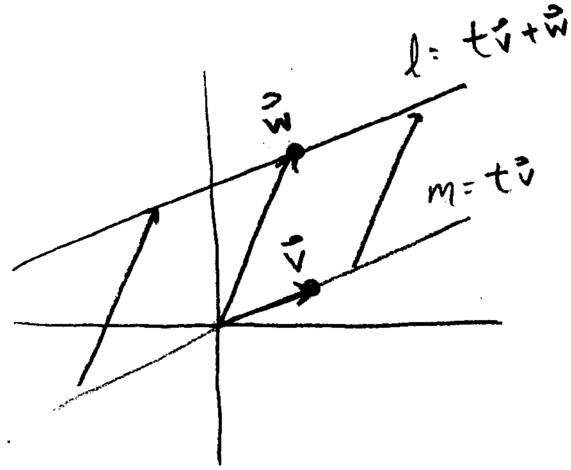
$$= g(t) - g(t) = 0.$$

Thus, $y - y_p$ solves $y_h' + a(t)\, y_h = 0$.

$$\Rightarrow y_h(t) = y(t) - y_p(t) \Rightarrow y(t) = y_h(t) + y_p(t) = C y_1(t) + y_p(t). \quad \square$$

<u>Remark</u>: <u>Any</u> sol'n $y_p(t)$ works!

Compare to parametrized lines:



* A line m through $\vec{0}$ can be written $t\vec{v}$, for some $\vec{v}$. This is a <u>vector space</u>.
* A line l parallel to $\vec{0}$ can be written $t\vec{v} + \vec{w}$ where $\vec{w}$ is <u>any</u> vector on l.

This is <u>not</u> a vector space, but it's close. (It's an <u>affine</u> space)

Examples:

(i) Solve $y' - 2y = 1$.

Homog: $y_h' - 2y_h = 0 \Rightarrow y_h(t) = Ce^{2t}$

Part.: Guess $y_p(t) = a$ (why will this work?)

Plug this back in: $0 - 2a = 1 \Rightarrow a = -\frac{1}{2}$.

Add: $y(t) = Ce^{2t} - \frac{1}{2}$.

(ii) Solve $y' - 2y = -4t^2$.

Homog: $y_h(t) = Ce^{2t}$

Part: Guess $y_p(t) = at^2 + bt + c$ (why?)

$$y_p'(t) = 2at + b$$

Plug back in: $(2at + b) - 2(at^2 + bt + c) = -4t^2$

$$(-2a)t^2 + (2a - 2b)t + (b - 2c) = -4t^2 + 0t + 0.$$

Equate coeffs:
$$\begin{cases} -2a = -4 \\ 2a - 2b = 0 \\ b - 2c = 0 \end{cases} \Rightarrow \begin{array}{l} a = 2 \\ b = 2 \\ c = 1 \end{array}$$

Thus, $y_p(t) = 2t^2 + 2t + 1$.

Add: $y(t) = Ce^{2t} + 2t^2 + 2t + 1$.

(iii) Solve $y' - 2y = 4e^{3t}$

Homog: $y_h(t) = Ce^{2t}$

Part: Guess $y_p(t) = ae^{3t}$ (why?)

$$y_p'(t) = 3ae^{3t}$$

Plug back in: $3ae^{3t} - 2ae^{3t} = 4e^{3t}$

$$ae^{3t} = 4e^{3t} \Rightarrow a = 4.$$

Thus, $y_p(t) = 4e^{3t}$ is a sol'n.

Add: $y(t) = Ce^{2t} + 4e^{3t}$.

(iv) Solve $y' - 2y = -13 \sin 3t$

Homog. $y_h(t) = Ce^{2t}$

Part. Guess $y_p(t) = a \cos 3t + b \sin 3t$

$$y_p'(t) = -3a \sin 3t + 3b \cos 3t$$

Plug back in: $(-3a \sin 3t + 3b \cos 3t) - 2(a \cos 3t + b \sin 3t) = -13 \sin 3t$

$$(3b - 2a) \cos 3t + (-3a - 2b) \sin 3t = 0 \cos 3t - 13 \sin 3t$$

Equate coeffs: $\begin{cases} 3b - 2a = 0 \\ -3a - 2b = -13 \end{cases} \Rightarrow \begin{array}{l} a = 3 \\ b = 2 \end{array}$

$\Rightarrow y_p(t) = 3 \cos 3t + 2 \sin 3t$.

Add: $y(t) = Ce^{2t} + 3 \cos 3t + 2 \sin 3t$.

Think: What could go wrong with this method?

Ex. (v) Solve $y' - 2y = e^{2t}$

Homog: $y(t) = Ce^{2t}$

Part: Guess $y_p(t) = ae^{2t}$, $\quad y_p'(t) = 2ae^{2t}$

Plug back in: $2ae^{2t} - 2(ae^{2t}) = e^{2t}$

$$0a = e^{2t} \quad \text{No such } a!$$

<u>Problem</u>: Our guess $y_p(t) = ae^{2t}$ actually solves the homogeneous eq'n, so it will <u>never</u> solve the inhomogeneous eq'n!

<u>Solution</u>: Guess $y_p(t) = a t e^{2t}$.

$$y_p'(t) = 2ate^{2t} + ae^{2t}$$

Plug back in: $(2ate^{2t} + ae^{2t}) - 2(ate^{2t}) = e^{2t}$

$$ae^{2t} = e^{2t} \Rightarrow a = 1 \Rightarrow y_p(t) = te^{2t}$$

<u>Add</u>: $y(t) = Ce^{2t} + te^{2t}$

<u>Remark</u>: This technique is called the <u>method of undetermined coefficients</u>.

It works as long as we can:

(i) Solve the homogeneous eq'n

(ii) Find a solution by inspection or educated guess.

If we can't do (i), we're hosed.

If we can't do (ii), then we have two alternative methods:

① Integrating factor

② Variation of parameters.

These work for more equations (as long as we can integrate $e^{\int a(t)} g(t)$), but they're more complicated.

They work "equally well," but Variation of parameters has a "built in check-point", so it's slightly better. Also, once we solve $y_h(t)$, we're "half-way there."

Variation of parameters: Solve $y' + a(t)y = g(t)$.

Step 1: Solve the related homogeneous eq'n, $y_h' + a(t)\, y_h = 0$.

let $y_h(t) = C\, y_1(t)$ be the sol'n.

Step 2: Assume the gen'l sol'n to the original equation is

$y(t) = v(t)\, y_1(t)$. Plug back in and find $v(t)$.

Example: Solve $y' + 2t\,y = t$.

Step 1: Solve $y_h' = -2t\, y_h$

$$\frac{dy_h}{dt} = -2t\, y_h \Rightarrow \int \frac{dy}{y} = -2\int t\, dt \Rightarrow \ln y = -t^2 + C$$

$$\Rightarrow y_h(t) = Ce^{-t^2}.$$

Step 2: Assume $y(t) = v(t)\, e^{-t^2}$

$y'(t) = v' e^{-t^2} - 2te^{-t^2} v$

check-point: If this doesn't cancel, something's wrong!

Plug in: $\left(v'e^{-t^2} - \cancel{2te^{-t^2}v}\right) + \cancel{2\left(tve^{-t^2}\right)} = t$

$v' = te^{t^2}$

$v = \int te^{t^2} dt = \frac{1}{2}e^{t^2} + C.$

"absorb" the $\frac{1}{2}$

$\Rightarrow y(t) = v(t)\, e^{-t^2} = \left(\frac{1}{2}e^{t^2} + C\right)e^{-t^2} = \frac{1}{2} + \frac{1}{2}Ce^{-t^2}$

$\Rightarrow y(t) = Ce^{-t^2} + \frac{1}{2}.$

Note: In retrospect, maybe we could have guessed $y_p(t) = \frac{1}{2}$!

$\boxed{2^{nd} \text{ order}}$ Consider $y'' + a(t)y' + b(t)y = g(t)$, $g(t) \neq 0$.

Define the <u>related homogeneous equation</u> to be $y_h'' + a(t)y_h' + b(t)y_h = 0$.

<u>Theorem</u>: let $y(t)$ be the general solution to $y'' + a(t)y' + b(t)y = g(t)$ and let $y_h(t)$ be the general solution to the related homogeneous equation, $y_h'' + a(t)y_h' + b(t)y_h = 0$. (So $y_h(t) = C_1 y_1(t) + C_2 y_2(t)$)

Then if $y_p(t)$ is <u>any</u> particular sol'n to the original ODE,

$y(t) = y_h(t) + y_p(t)$

<u>Proof</u>: HW 3 (show that $y - y_p$ solves the homogeneous eq'n)

Thus, to solve a 2^{nd} order linear inhomogeneous ODE, take the following steps:

1. Solve the homog. eq'n: $y_h'' + a(t)y_h' + b(t)y_h = 0$.
2. Find <u>any</u> particular sol'n $y_p(t)$ to the original eq'n
3. Add these together: $y(t) = y_h(t) + y_p(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$.

<u>Examples</u>:

(i) Solve $y'' + 3y' + 2y = 1$.

<u>Homog</u>: $y_h(t) = C_1 e^{-t} + C_2 e^{-2t}$ (exercise)

<u>Part</u>: Guess $y_p(t) = a$ Note that $a = \frac{1}{2}$ works (why?)

<u>Add</u>: $y(t) = C_1 e^{-t} + C_2 e^{-2t} + \frac{1}{2}$

(ii) Solve $y'' + 3y' + 2y = -4t^2$

Homog: $y_h(t) = C_1 e^{-t} + C_2 e^{-2t}$

Part: Guess $y_p(t) = at^2 + bt + c$.

Plug back in and solve for a, b, c. (Exercise)

Add: $y(t) = C_1 e^{-t} + C_2 e^{-2t} + y_p(t)$

(iii) Solve $y'' + 3y' + 2y' = 4e^{3t}$

Homog: $y_h(t) = C_1 e^{-t} + C_2 e^{-2t}$

Part: Guess $y_p(t) = ae^{3t}$. Solve for a. (Exercise)

Add: $y(t) = C_1 e^{-t} + C_2 e^{-2t} + y_p(t)$.

Summary.

The solutions to linear homogeneous ODE's form a vector space.

e.g., $\{C_1 y_1(t) \mid C_1 \in \mathbb{R}\}$, or $\{C_1 y_1(t) + C_2 y_2(t) \mid C_1, C_2 \in \mathbb{R}\}$.

The solutions to linear, inhomogeneous ODEs have the form

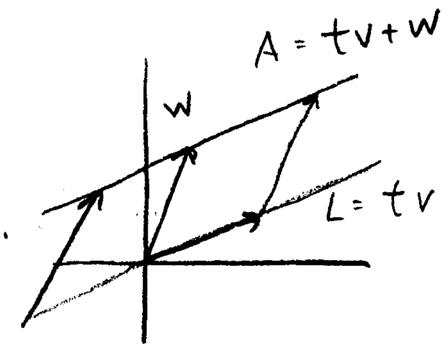
e.g., $\{C_1 y_1(t) + y_p(t) \mid C_1 \in \mathbb{R}\}$, or $\{C_1 y_1(t) + C_2 y_2(t) + y_p(t) \mid C_1, C_2 \in \mathbb{R}\}$.

These are not vector spaces, but they are "close". They are called affine spaces.

Think: Line, plane, etc., but not through the origin $\vec{0}$.

<u>Def:</u> An <u>affine space</u> is a set A (of vectors) and a set F (of scalars) such that for some vector $w \in A$, the set $A - w := \{ v - w \mid v \in V \}$ is a vector space over F.

<u>Examples</u>:



(1a) Fix a vector $v \in \mathbb{R}^n$.

The set $L = \{ tv \mid t \in \mathbb{R} \}$ is <u>vector space</u>. it is a line through $\vec{0}$.

Now, fix $w \notin L$. The set $A = \{ tv + w \mid t \in \mathbb{R} \}$ is <u>not</u> a vector space. (it's a line not through $\vec{0}$)

But the set $A - w := \{ (tv + w) - w \mid t \in \mathbb{R} \} = \{ tv \mid t \in \mathbb{R} \} = L$ is a vector space. So A is an <u>affine space</u>.

<u>Note</u>: We can write the affine space A as

$$A = \{ tv + w \mid t \in \mathbb{R} \} \text{ for } \underline{\text{any}} \ w \in A.$$

i.e, if we <u>parametrize</u> L by $\ell(t) = tv$, then we can parametrize A by $a(t) = \ell(t) + w$, for any fixed $w \in A$.

(1b) Suppose $y_1(t) \neq 0$ solves $y_h' + a(t)\, y_h = 0$.

The solution space $L = \{ C y_1(t) \mid C_1 \in \mathbb{R} \}$ is a vector space.

Now, suppose $y_p(t)$ solves $y' + a(t)\,y = g(t)$, where $g(t) \neq 0$.

The set $A := \{ C_1 y_1(t) + y_p(t) \mid C_1 \in \mathbb{R} \}$ is <u>not</u> a vector space.

But the set $A - y_p(t) := \{ (C_1 y_1(t) + y_p(t)) - y_p(t) \mid C_1 \in \mathbb{R} \} = L$

is a vector space. So A is an <u>affine space</u>.

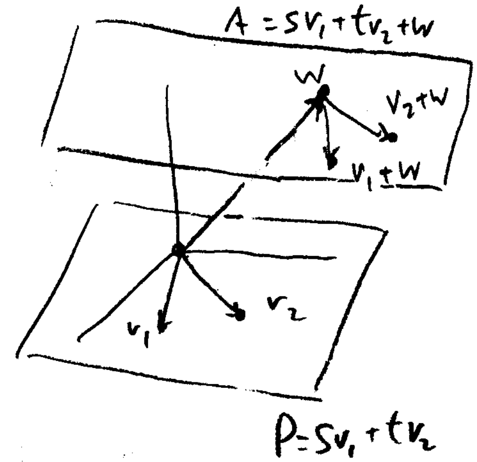
<u>Note</u>: We can write the affine space M as

$M = \{ C_1 y_1(t) + y_p(t) \mid C_1 \in \mathbb{R} \}$ for <u>any</u> $w \in A$, i.e., if

we parametrize L by $y_h(t) = C_1 y_1(t)$, then we can

parametrize A by $y_h(t) = C_1 y_1(t) + y_p(t)$, for <u>any</u> $y_p(t)$.

(2a) Fix non-colinear vectors $v_1, v_2 \in \mathbb{R}^n$.



The set $P = \{ s v_1 + t v_2 \mid s, t \in \mathbb{R} \}$ is a

<u>vector space</u> (a plane through $\vec{0}$).

Now, fix $w \notin P$. The set

$A = \{ s v_1 + t v_2 + w \mid s, t \in \mathbb{R} \}$ is <u>not</u>

a vector space, but it's an <u>affine space</u>.

We can parametrize P by $p(s,t) = s v_1 + t v_2$, $s, t \in \mathbb{R}$

We can parametrize Q by $q(s,t) = s v_1 + t v_2 + w$, $s, t \in \mathbb{R}$,

valid for any fixed $w \in Q$.

(2b) Suppose $y_1(t)$ and $y_2(t)$ are linearly independent solutions to

$y_h'' + a(t)\,y_h' + b(t)\,y_h = 0$.

The solution space $\mathbb{P} = \{ C_1 y_1(t) + C_2 y_2(t) \mid C_1, C_2 \in \mathbb{R} \}$ is a vector space.

Now, suppose $y_p(t)$ solves $y'' + a(t) y' + b(t) y = g(t)$, $g(t) \neq 0$.

The set $A = \{ C_1 y_1(t) + C_2 y_2(t) + y_p(t) \mid C_1, C_2 \in \mathbb{R} \}$ is not a vector space. But it is an affine space.

We can <u>parametrize</u> $\mathbb{P}$ by $y_h(t) = C_1 y_1(t) + C_2 y_2(t)$, $C_1, C_2 \in \mathbb{R}$, and parametrize A by $y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$, valid for <u>any</u> particular solution $y_p(t)$.

* In summary, when we do a <u>change of variables</u>,

e.g., let $(y_1, y_2) = (x_1 - w_1, x - w_2)$

or $y_h(t) = y(t) - y_p(t)$,

we're usually just making an inhomogeneous equation into a homogeneous one.