3. Power series solutions to ODE's.

We've seen how to solve 2^{nd} order constant-coefficient ODE's.

Now, consider the following: $t^2y'' + ty' - y = 0.$

What's a good guess? Try $y(t) = t^r$ (why?)

$$y'(t) = rt^{r-1}, \quad y''(t) = r(r-1)t^{r-2}.$$

Plug back in: $t^2y'' + ty' - y = t^2 r(r-1)t^{r-2} + tr t^{r-1} - t^r = 0$

$$\Rightarrow t^r(r^2-1) = 0 \quad \Rightarrow r = \pm 1.$$

We have 2 solutions: $y_1(t) = t, \quad y_2(t) = t^{-1}.$

The general solution is thus $\boxed{y(t) = C_1 t + C_2 t^{-1}}$

Suppose r is complex, and $r = a \pm bi.$

Then $y_1(t) = t^{a+bi}, \quad y_2(t) = t^{a-bi}$ are solutions.

<u>Note</u>: $t^{a+bi} = t^a t^{bi} = t^a (e^{\ln t})^{bi} = t^a e^{(b \ln t)i}$

$$= t^a[\cos(b\ln t) + i \sin(b \ln t)] = t^a \cos(b \ln t) + i\, t^a \sin(b \ln t)$$

$\Rightarrow t^a \cos(b \ln t)$ & $t^a \sin(b \ln t)$ are solutions (Why?)

Thus, the general solution is $\boxed{y(t) = C_1 t^a \cos(b \ln t) + C_2 t^a \sin(b \ln t)}$

<u>Remark</u>: We could have written the general solution as

$$y(t) = C_1 t^{a+bi} + C_2 t^{a-bi}$$

<u>Remark</u>: If r is a <u>repeated root</u>, then $y_1(t) = t^r$, and we need to look for a solution $y_2(t) = v(t)\, t^r$. We'll get $v(t) = \ln t$, and so $\boxed{y(t) = C_1 t^r + C_2 t^r \ln t}$ will be the general solution. (Work omitted; it's slightly tedious.)

<u>Let's make things harder.</u>

Consider $y'' - 4t y' + 12 y = 0.$

What should we guess the solution will be?

<u>Note</u>: $y(t) = t^r$ won't work!

Because if $y = t^r$, $y' = r t^{r-1}$, $y'' = r(r-1) t^{r-2}$,

then $y'' - 4ty' + 12y = \underbrace{r(r-1)}_{\neq 0} t^{r-2} + \underbrace{(12-4r)}_{\neq 0} t^r = 0$ no sol'n for r!

Maybe try $y(t) = a t^r + b t^{r-2}$?

Then, we'll get $(\qquad) t^{r-4} + (\qquad) t^{r-2} + (\qquad) t^r = 0$

This will give us a solution, since we have 3 equations (set coeffs to 0) and 3 unknowns (a, b, r).

But it'll only give us <u>one</u> sol'n (up to scalars).

<u>Better method</u>: Most "nice" functions have a Taylor series expansion, so let's look for a sol'n of that form.

Assume $y(t) = \sum_{n=0}^{\infty} a_n t^n$.

$$y'(t) = \sum_{n=0}^{\infty} n a_n t^{n-1}, \quad y''(t) = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2}$$

Plug back in: $$\underbrace{\sum_{n=0}^{\infty} n(n-1) a_n t^{n-2}}_{\text{re-write this so we can combine terms.}} - 4\sum_{n=0}^{\infty} n a_n t^n + 12 \sum_{n=0}^{\infty} a_n t^n = 0 \qquad (*)$$

let $m = n-2$: (so $n = m+2$) $$\sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} = \sum_{m=-2}^{\infty} (m+2)(m+1) a_{m+2} t^m$$

$$(\text{why?}) \quad = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} t^m$$

We've shifted the indices without changing the series!

Observe (for example / motivation):

If $f(t) = \sum_{n=0}^{\infty} t^n = 1 + t + t^2 + t^3 + t^4 + t^5 + \ldots$

then $f'(t) = \boxed{\sum_{n=0}^{\infty} n t^{n-1}} = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + \ldots = \boxed{\sum_{n=0}^{\infty} (n+1) t^n}$

Now, switch back to using n (from m):

$(*)$ becomes $$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n - 4 \sum_{n=0}^{\infty} n a_n t^n + 12 \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{n=0}^{\infty} \Big[\underbrace{(n+2)(n+1) a_{n+2} + (12-4n) a_n}_{\text{must be } = 0.}\Big] t^n = 0$$

$\Rightarrow (n+2)(n+1) a_{n+2} + (12-4n) a_n = 0$ for all n

$\Rightarrow \boxed{a_{n+2} = \dfrac{4(n-3)}{(n+2)(n+1)} a_n}$ This is a recurrence relation.

Note: $y(0) = a_0$ and $y'(0) = a_1$

Choose any a_0. All of the even a_n's are determined.

Choose any a_1. All of the odd a_n's are determined.

Thus, we have a 2-parameter infinite family of solutions:

$$y_1(t) = \sum_{n=0}^{\infty} a_{2n} t^{2n} \quad \text{(for some fixed } a_0\text{)}$$

$$y_2(t) = \sum_{n=0}^{\infty} a_{2n+1} t^{2n} \quad \text{(for some fixed } a_1\text{)}$$

Then $\{y_1(t), y_2(t)\}$ is a basis for the sol'n space!

i.e, the general sol'n is $y(t) = \sum_{n=0}^{\infty} a_n t^n = \underbrace{C_1 y_1(t)}_{\text{even terms}} + \underbrace{C_2 y_2(t)}_{\text{odd terms}}$

Let's compute the first few terms (in terms of a_0 & a_1).

$$a_2 = -\frac{12}{2} a_0 = -6 a_0 \qquad\qquad a_3 = \frac{-8}{3} a_1 = -\frac{4}{3} a_1$$

$$a_4 = \frac{-4}{4 \cdot 3} a_2 = \frac{(-4)(-12)}{4!} a_0 = 2 a_0 \qquad\qquad a_5 = 0$$

$$a_6 = \frac{4(-4)(-12)}{6!} a_0 = \frac{4}{15} a_0 \qquad\qquad a_7 = 0$$

$$\vdots \qquad\qquad \vdots$$

$$a_n = \frac{(4 \cdot 0 - 12) - (4 \cdot 2 - 12)(4 \cdot 4 - 12) \cdots [4(n-2) - 12]}{n!}$$

<u>Remark</u>: If $a_0 = 0$, then $y(t) = a_1 t + a_3 t^3$

$$= a_1 t - \frac{4}{3} a_1 t^3 = \boxed{\overbrace{a_1\left(t - \tfrac{4}{3}t^3\right)}^{y_2(t)}}$$

This is the only <u>polynomial</u> solution up to scalars (why?)

<u>Summary</u>: To solve $y'' - 4ty' + 12y = 0$, we

* Assumed $y(t) = \sum_{n=0}^{\infty} a_n t^n$
* Plugged back into the ODE, combined into a single sum $\sum_{n=0}^{\infty} [\quad] t^n = 0$ (need to "shift indices").
* Set coefficients to 0 to get a recurrence: $a_{n+2} = (\quad) a_n$.

<u>Quick review of power series</u>:

<u>Def</u>: A <u>power series</u> centered at t_0 is a series of the form $\sum_{n=0}^{\infty} a_n (t-t_0)^n = \lim_{N\to\infty} \underbrace{\sum_{n=0}^{N} a_n (t-t_0)^n}_{\text{"partial sum"}}$

* Henceforth, we will only consider power series centered at $t_0 = 0$, i.e., $y(t) = \sum_{n=0}^{\infty} a_n t^n$.

A power series <u>converges</u> at t if the sequence of partial sums converges. Otherwise it <u>diverges</u>.

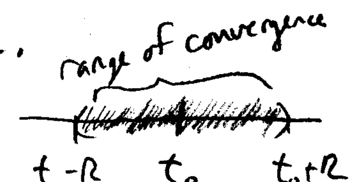
<u>Example</u>: $\lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!} t^n$ converges to e^t for all t

Non-example: $\lim_{N\to\infty} \sum_{n=0}^{N} (-1)^n t^n$ diverges for $t=1$ because the sequence of partial sums $\sum_{n=0}^{N} (-1)^n 1^n = 1-1+1-1 \pm \cdots$ is $1, 0, 1, 0, 1, 0 \ldots$ (does not converge.)

Key point: Sometimes a series won't converge everywhere.

Example: $y(t) = \sum_{n=0}^{\infty} t^n$:
- Converges to $\frac{1}{1-t}$ if $|t|<1$
- Diverges if $|t|=1$.

Def: The radius of convergence is the largest number R such that if $|t-t_0|<R$, then $\sum_{n=0}^{\infty} a_n (t-t_0)^n$ converges. If it converges for all t, we say $R=\infty$.



Examples: $y(t) = \sum_{n=0}^{\infty} t^n$ has $R=1$

$y(t) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n$ has $R=\infty$ (it converges to e^t.)

The Ratio Test (for computing R)

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}, \text{ if this limit exists.}$$

Examples (1) Taylor series for $\ln(t+1) = \sum_{n=1}^{\infty} (-1)^{n+1} t^n = t - t^2 + t^3 - t^4 \pm \cdots$

So, $|a_n| = 1$ for all $n \geq 1$ $\Rightarrow$ $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \frac{1}{1} \Rightarrow R = 1$.

(2) $y(t) = \sum_{n=0}^{\infty} \frac{1}{3^n} t^n \quad a_n = \frac{1}{3^n}, \quad \lim_{n\to\infty} |1/3^n| / |1/3^{n+1}| = 3 \Rightarrow R = 3.$

(3) $e^t = \sum_{n=0}^{\infty} \frac{1}{n!} t^n, \quad a_n = \frac{1}{n!} \quad \lim_{n\to\infty} |1/n!| / |1/(n+1)!| = n+1 \to \infty$

$\Rightarrow \boxed{R = \infty}$

<u>Regular vs. singular points of ODEs:</u>

<u>Def</u>. A function $f(t)$ is <u>real analytic</u> at t_0 if

$f(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^n$ for some $R > 0$.

i.e, real analytic $\Leftrightarrow$ has a power series.

<u>Def</u>: Consider the ODE $y'' + P(t)\, y' + Q(t)\, y = 0$.

* The point t_0 is an <u>ordinary point</u> if $P(t)$ & $Q(t)$ are real analytic at t_0.
* If t_0 is not ordinary, then it is a <u>singular point</u>.
  - If t_0 is singular, then it is <u>regular</u> if $(t-t_0)P(t)$ <u>and</u> $(t-t_0)^2 Q(t)$ are real analytic at t_0.

<u>Remark</u>: In most cases, "real analytic" just means "defined,"

e.g., $\frac{1}{t}$ is real analytic at $t_0 = 1$ but not at $t_0 = 0$.

Why we care:

<u>Theorem of Frobenius</u>: Consider an ODE $y'' + P(t)y' + Q(t)y = f(t)$, where $f(t)$ is real analytic at t_0.

- If t_0 is an <u>ordinary point</u>, and P, Q, f have radii of convergence R_P, R_Q, R_F, respectively, then there is a <u>power series solution</u> $y(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^n$, with $R = \min\{R_P, R_Q, R_F\}$.
- If t_0 is a <u>regular singular point</u>, and $(t-t_0)P(t)$, $(t-t_0)Q(t)$, and $f(t)$ are analytic with radii of convergence R_P, R_Q, R_F, then there is a <u>generalized power series solution</u> $y(t) = (t-t_0)^r \sum_{n=0}^{\infty} a_n (t-t_0)^n$ for some constant r. (possibly a fraction, or even complex).
- If t_0 is an irregular singular point, we're out of luck.

<u>Example</u>: Consider $y'' + t^2 y - 4y = 0$. Here, $P(t) = t^2$, $Q(t) = -4$ are both analytic for all t_0, with radii of convergence $R = \infty$. Thus, by Frobenius, there is a solution $y(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^n$, valid for all t.

<u>Example</u>: $(t^2-1)y'' + ty' - p^2 y = 0$.

Write as $y'' - \frac{t}{1-t^2} y' + \frac{p^2}{1-t^2} y = 0$

$$Q(t) = \frac{1}{1-t^2} = \sum_{n=0}^{\infty} (t^2)^n = \sum_{n=0}^{\infty} t^{2n} = 1 + t^2 + t^4 + t^6 + \dots$$

$$P(t) = \frac{-t}{1-t^2} = -t \sum_{n=0}^{\infty} (t^2)^n = \sum_{n=0}^{\infty} -t^{2n+1} = -t - t^3 - t^5 - t^7 - \dots$$

By the ratio test, $R_P = R_Q = 1$.

Thus, by Frobenius, there is a solution $y(t) = \sum_{n=0}^{\infty} a_n t^n$ with $R = 1$.

This ODE is called Chebyshev's equation.

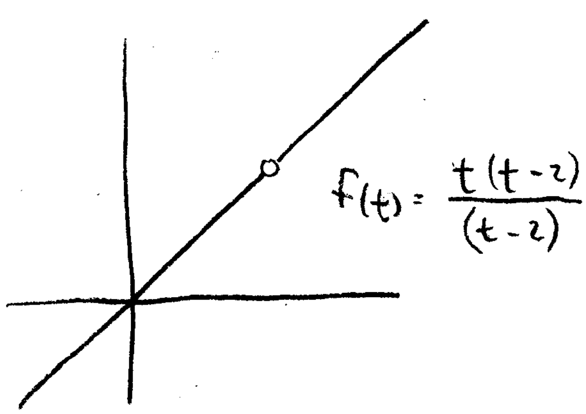
<u>Example</u>: $t^5 y'' + y' + y = 0$.

Write as $y'' + \frac{1}{t^5} y' + \frac{1}{t^5} y = 0$. $P(t) = \frac{1}{t^5}$, $Q(t) = \frac{1}{t^5}$.

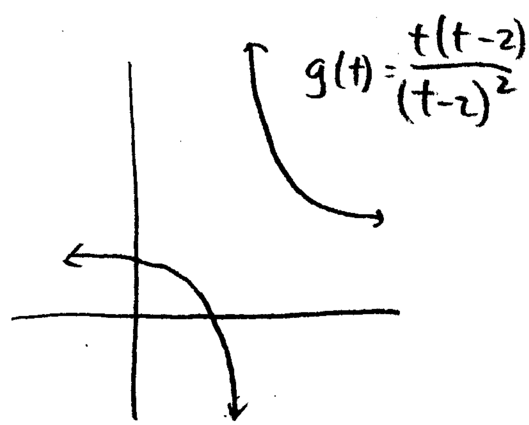
$t_0 = 0$ is an irregular singular point, since $t P(t) = \frac{1}{t^4}$ isn't defined at $t_0 = 0$.

Frobenius does <u>not</u> guarantee a solution of the form $y(t) = \sum_{n=0}^{\infty} a_n t^n$. But we might be able to find one of the form $\sum_{n=0}^{\infty} a_n (t-1)^n$ (because $t_0 = 1$ is regular).

<u>Analogy</u> of regular vs. irregular



This singularity is "fixable."



This singularity is "unfixable."

Example: Solve $2ty'' + y' + y = 0$.

Write as $y'' + P(t)y' + Q(t)y = 0$, $\quad P(t) = \frac{1}{2t}$, $\quad Q(t) = \frac{1}{2t}$

$t_0 = 0$ is a regular singular point, since $tP(t) = \frac{1}{2}$ and $t^2 Q(t) = \frac{1}{2}t$ are real analytic (ie, defined).

By Frobenius, there is a solution of the form

$$y(t) = t^r \sum_{n=0}^{\infty} a_n t^r = \sum_{n=0}^{\infty} a_n t^{n+r}.$$

We'll find it the same way as before.

$$y'(t) = \sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1}, \qquad y''(t) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r-2}$$

Plug back into the ODE:

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n t^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1} + \sum_{n=0}^{\infty} a_n t^{n+r} = 0$$

$$= t^r \left[\sum_{n=0}^{\infty} (2n+2r-1)(n+r) a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n \right] = 0$$

Shift indices up by 1 (let $m = n-1$ or just do in your head).

$$= \cancel{t^r} \left[\sum_{n=-1}^{\infty} (2n+2r+1)(n+r+1) a_{n+1} t^n + \sum_{n=0}^{\infty} a_n t^n \right] = 0$$

one extra term!

$$= \underbrace{(2r-1) r a_0}_{\text{set } = 0} t^{-1} + \sum_{n=0}^{\infty} \underbrace{\left[(2n+2r+1)(n+r+1) a_{n+1} + a_n \right]}_{\text{set } = 0} t^n = 0$$

$\Downarrow$

$(2r-1)r = 0$ "indicial equation"

$\Downarrow$

$$\boxed{r = 0 \text{ or } r = \tfrac{1}{2}}$$

$\Downarrow$

$$\boxed{a_{n+1} = \frac{-1}{(2n+2r+1)(n+r+1)} a_n}$$

"recurrence relation"

We now have two generalized power series solutions:

$\underline{r=0}$: $y_0(t) = \sum_{n=0}^{\infty} a_n t^n$, $a_{n+1} = \frac{-1}{(2n+1)(n+1)} a_n$

$\underline{r=\frac{1}{2}}$: $y_{1/2}(t) = \sqrt{t} \sum_{n=0}^{\infty} a_n t^n$, $a_{n+1} = \frac{-1}{(2n+2)(n+3/2)} a_n$

<u>Remark</u>: This time, choosing a_0 determines <u>every</u> a_n, but we still have 2 linearly independent solutions: $\{y_0(t), y_{1/2}(t)\}$ is a basis for the solution space.

The general solution is $y(t) = C_1 y_0(t) + C_2 y_{1/2}(t)$.

* The power series method really does come up in practice!

- <u>Hermite's diff. eq'n</u>: $y'' - 2ty' + 2py = 0$.
  Used for modeling simple harmonic oscillators in quantum mechanics.
- <u>Legendre's diff eq'n</u>: $(1-t^2)y'' - 2ty' + p(p+1)y = 0$.
  Used for modeling spherically symmetric potentials in theory of Newtonian gravitation, and in electricity & magnetism (E & M).
- <u>Bessel's equation</u>: $t^2 y'' + t y' + (t^2 - p^2) y = 0$.
  Used for analyzing vibrations of a circular drum.
- <u>Chebyshev's equation</u>: $(1-t^2)y'' - ty' + p^2 y = 0$.

<u>Remark</u>: Solving <u>inhomogeneous</u> ODE's of this form only requires additionally finding some particular sol'n, $y_p(t)$. But this can be tricky. However, the <u>variation of parameters</u> method for 2^{nd} order equations will indeed work. It's complicated, we won't do it here.